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Congressional Apportionment: A Liberal Arts Perspective

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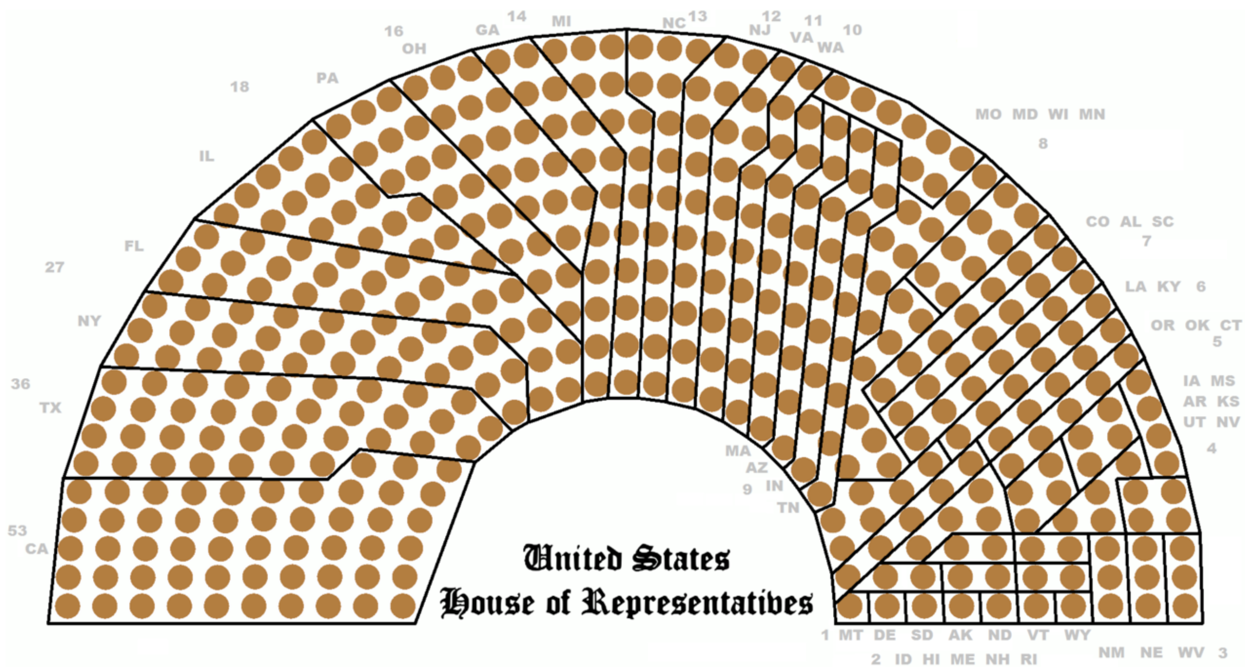

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Preface

Congressional apportionment of the United States House of Representatives is a popular topic in several liberal arts or general education mathematics textbooks. This work is produced in the belief that any illustrative applications of mathematics should be presented at the same high level as the mathematics. Since congressional apportionment is such a mainstay as a principle example of a mathematical, fixed resource problem, we believe that a more thorough presentation of the topic as it unfolded in American history is in order. Accordingly we dedicate five individual topics under the general heading of apportionment: averages and rounding, the basic divisor method, the quota method, the modified divisor method, and priority computational techniques.

This work is intended as an open source endeavor. Comments, suggestions, critique, etc., are most welcome. The author is especially grateful to Professors Dale Oliver, Adam Falk, and Tim Lauck at Humboldt State University, and to Professors Dan Munton and John Martin at Santa Rosa Junior College for their collaborations and guest lecture invitations. I am also grateful for reviews from my dear wife Carolyn, and to friends Mickie York, Grace Kimura, and George Robinson. Special thanks goes to Professor David Lippman, Pierce College, Fort Steilacoom, Washington, for his encouragement and invitation to expand on his work on apportionment in the open-source text, *Math and Society: A survey of mathematics for the liberal arts major*, 2102. The sections in this work on quota method and the modified divisor methods are built on the foundation of the Apportionment chapter *Math and Society*.

For additional open-resource materials, please visit http://www.nia977.wix.com/drbcap.

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https://commons.wikimedia.org/wiki/File:USHouseStructure2012-2022_SeatsByState.png.

Section 1: An Average Lesson

An important aspect of education is to learn how to do **research**. A research project investigates a general field of interest along with its components. Research also includes the process of asking questions of interest and obtaining answers to some of those questions. In today's world the general public often just focuses the concept of research on getting answers. However, much preparation must be done before one can get useful answers. Research encompasses the entire process.

1. Background

In conducting research one often comes across classic research problems. For example, how can one say something informative about a group when the individuals in the group are all different? This problem is often called the **Problem of Variation** or the **Problem of Diversity**. The word "problem" here is to be taken in a positive sense meaning a situation or reality. One should not take the word "problem" here in a negative sense meaning something disgusting or something to be avoided.

Along with classic research problems come classic solutions; for example, **denial**. One way to face the problem of variation is to tell yourself, "what variation?" Ask yourself, "What would all the members of the group look like if they all looked the same?" This is precisely the kind of thinking behind the statistical concept of **average** and embodied in the English word **typical**. By appealing to a "typical" member of a group you can begin to talk about the members of a group in an organic, living sense. Of course denial is unrealistic if you stop there. But it can be an informative and useful beginning. To solve a problem (problem in the negative sense), ask yourself what the world would be like without the problem. For example, imagine a world without war, disease, or poverty.

Imagine now a research project where you have a group of interest and also a question of interest that can be posed to each member of the group. Statisticians call such questions **random variables** because the members of the group give different answers to the question, hence the answers vary from member to member. The answers are random in that you do not know in advance the answer when the next member of the group presents itself. For example, my group of interest could be all the students enrolled in the current semester at Humboldt State University. Questions of interest (random variables) could include: What is your name? What level student are you (freshman, sophomore, junior, senior, graduate)? How old are you? What is your serum tetrahydrocannabinol level?

Some questions of interest must be answered with a number obtained by counting or measuring something. Statisticians call such questions **quantitative random variables**. Statisticians calculate a mean in order to obtain an average for a list of numbers obtained from such variables. A **mean** of a set of numbers is an average for those numbers calculated in a way that preserves some rationale for those numbers. In this way, rather than being boggled by a bewildering array of numbers, one bypasses the variation and simply refers to this typical value. For example, HSU's website says that the average cost for a resident student at HSU is $23,366.[1] Although costs differ from student to student, this average is a helpful reference when planning finances.

[1] https://www.humboldt.edu/cost. Accessed September 2015.

2. Means

We now formalize the computational aspects of a mean. Recall that a mean of a set of numbers is an average calculated in a way that preserves some rationale for those numbers. Since numbers may be used for a variety of different purposes, accordingly there are a variety of different means. The three most common means are the arithmetic mean, the geometric mean, and the harmonic mean. We begin with the simplest kind of group, a group with only two members. So our current quest is: given two positive numbers *a* and *b*, how and why do we calculate the various means?

Arithmetic Mean

The **arithmetic mean** is the ordinary average of two numbers that is usually taught in the fifth grade: add the two numbers together and divide by 2. For example, the arithmetic mean of 4 and 16 is calculated by (4 + 16)/2 = 10. We denote the arithmetic mean (AM) of 4 and 16 by AM(4,16) = 10. In general, suppose *a* and *b* are two positive numbers. Then, the arithmetic mean of *a* and *b* is given by

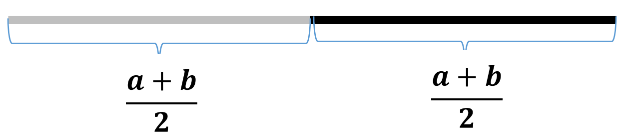
$$\text{AM}(a,b) = \frac{(a+b)}{2}$$

The arithmetic mean is an additive mean. Think of *a* and *b* as the length of two boards that we join linearly to make a longer board. The combined length is *a* + *b*. The thing we want to preserve is the combined length of the boards. Now consider making a final board with two components each having the same length. How long should each component be so that the result, a final board of length *a* + *b*, is preserved? For this rationale, each component needs to have length (*a* + *b*)/2; i.e., the arithmetic mean. Voila! Same result, but this time without the variation in the components.

Hence, the general rationale for the arithmetic mean is as follows. Suppose I have two different positive numbers and add them together to get their sum (notice the variation in the numbers you are adding together).

Total length *a* + *b*.

Now, remove the variation. What would you use if both numbers were replaced by the same quantity but still gave the same sum? Answer: replace *a* and *b* by their arithmetic mean.



Total length *a* + *b*.

Voila! Same result without the variation in the components.

Geometric Mean

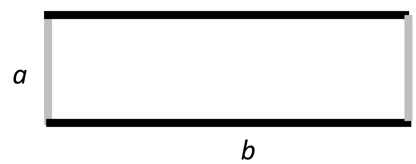
The **geometric mean** (GM) views the numbers *a* and *b* multiplicatively instead of additively. The geometric mean of two numbers is obtained by multiplying them together and then taking the square root of the resulting product. For example, GM(4,16) = $\sqrt{4 \times 16}$ = $\sqrt{64}$ = 8. In general, the geometric mean of two positive numbers *a* and *b* is given by

$$\text{GM}(a,b) = \sqrt{a \times b}$$

The obvious question is, who would want to do such a thing? The thinking behind the geometric mean is similar to the thinking behind the arithmetic mean, but the geometric mean focuses on multiplication rather than addition. When I multiply 4x16 = 64, I notice that I am multiplying two different numbers to get the answer of 64. What if I wanted to replace the two different numbers by a single number to get the same answer? Hence, ?x? = 64. To this end we need the square root of 64, which is 8.

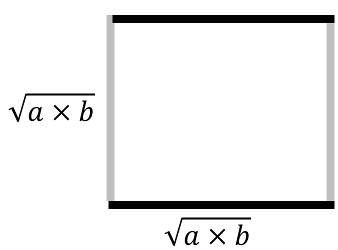
Suppose I made a rectangular laboratory space that is 4 ft. by 16 ft. The resulting working space is obtained by multiplication yielding a working area of 64 square feet. So, what would be the dimensions of the room which had the same work space, 64 square feet, but sides of equal length? For this rationale, we need the geometric mean. The arithmetic mean will not work.

Hence, the general rationale for the geometric mean is as follows. Suppose I have two different positive numbers and multiply them to get their product (notice the variation in the numbers you are multiplying together).



Total area $a \times b$.

Now, remove the variation. What would you use if both numbers were replaced by the same quantity but still gave the same product? Answer: replace *a* and *b* by their geometric mean.



Total area $a \times b$.

Voila! Same result without the variation in the components.

One can obtain a variety of applications of the geometric mean by googling "applications of the geometric mean," especially "applications of the geometric mean in business."

Harmonic Mean

An understanding of the harmonic mean begins with a classic formula from elementary school: distance = rate x time (in symbols, $d = rt$). For example, if I travel 240 miles in 6 hours, then my average speed is 40 miles per hour. Of course this does not mean that the car was in cruise control and you traveled 40 miles per hour the entire trip. What is so amazing about this averaging formula is that you don't even know what speeds are being "averaged!"

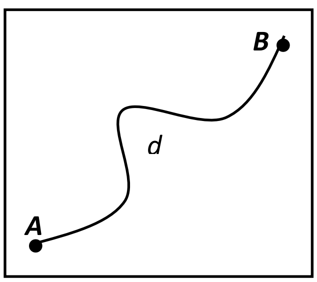
Now suppose you make the return trip along the same route but it takes you 4 hours. Your average speed for the return trip is (240 miles)/(4 hours) = 60 miles per hour. So the arrival trip was made at 40 mph and the return trip at 60 mph. What was your average speed for the entire trip? How would you average 40 and 60 to obtain your average trip speed? To answer these questions, we need to determine what rationale of the numbers we want to preserve. For an average total trip speed, the total distance is 480 miles (arrival plus return distances). The total time for the trip is 10 hours. Hence, the average speed for a 480 mile trip in 10 hours is 48 miles per hour. But, AM(40,60) = 50, so the arithmetic mean will not do. Also, GM(40,60) = $\sqrt{40 \times 60}$ = $\sqrt{2400}$ = 48.98979⋯, so the geometric mean won't do either. We need the harmonic mean.

The **harmonic mean** (HM) of two positive numbers *a* and *b* is given by

$$\text{HM}(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} = \frac{2 \times a \times b}{a+b} = \frac{a \times b}{(a+b)/2} = \frac{(\text{GM})^2}{\text{AM}} \quad (*)$$

Hence, the harmonic mean is a hybrid mean being equivalent to the square of the geometric mean divided by the arithmetic mean. In our example, HM(40,60) = 48.

A formal explanation for the harmonic mean formula in (*) may be given as follows. Suppose we want to make a round trip from *A* to *B* with distance *d* between the two points. The total round-trip distance is then 2*d*.



Suppose that the time for the arrival trip from *A* to *B* is t_1. Since, in general, $d = rt$, then also $r = d/t$ and $t = d/r$. So, the arrival trip speed, r_1, is calculated by $r_1 = d/t_1$. Thus, $t_1 = d/r_1$.

Similarly, suppose that the time for the return trip is t_2. Then the return speed, r_2, is calculated by $r_2 = d/t_2$. Thus, $t_2 = d/r_2$.

Now let *r* be the average speed and *t* be the total time for the whole round-trip. The round-trip distance is 2*d* and the round-trip time is given by $t = t_1 + t_2$. Then, for the entire round-trip, we have:

$$2d = rt$$

$$\Leftrightarrow \quad 2d = r(t_1 + t_2) \qquad \text{substitute: } t = t_1 + t_2$$

$$\Leftrightarrow \quad 2d = r\left(\frac{d}{r_1} + \frac{d}{r_2}\right) \qquad \text{substitute: } t_1 = d/r_1 \text{ and } t_2 = d/r_2$$

$$\Leftrightarrow \quad 2 = r\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \qquad \text{cancel through by } d$$

$$\Leftrightarrow \quad r = \frac{2}{\frac{1}{r_1}+\frac{1}{r_2}} \qquad \text{algebra}$$

$$\Leftrightarrow \quad r = \text{HM}(r_1, r_2) \qquad \text{definition of harmonic mean}$$

We conclude that the average speed for the round-trip is the harmonic mean of the arrival trip speed and the return trip speed.

3. Averaging

Suppose we are given two positive numbers, *a* and *b*. We want to report an average. What are our options? Here are five options that have applications. Only context and rationale can determine which option to use.

min(*a*,*b*)	This is the minimum of *a* and *b*. It is the "at least" option.
max(*a*,*b*)	This is the maximum of *a* and *b*. It is the "at most" option.
AM(*a*,*b*)	The arithmetic mean.
GM(*a*,*b*)	The geometric mean.
HM(*a*,*b*)	The harmonic mean.

The following interesting **mean inequality** relationship always occurs for positive numbers where $a < b$.

$$a = \min(a,b) < \mathrm{HM}(a,b) < \mathrm{GM}(a,b) < \mathrm{AM}(a,b) < \max(a,b) = b$$

For example, consider 4 and 16. Then,

$$4 = \min(4,16) < \mathrm{HM}(4,16) = 6.4 < \mathrm{GM}(4,16) = 8 < \mathrm{AM}(4,16) = 10 < \max(4,16) = 16$$

Further, the only way that two of the quantities can be equal happens if and only if the two numbers *a* and *b* were the same to begin with.

4. Extending the Means

Thus far we have provided mechanisms for calculating the AM, GM, and HM of two *positive* numbers. The formulas may be extended to include the special case that $a = 0$ and $b > 0$.

The formula for the arithmetic mean applies immediately:

$$\mathrm{AM}(0,b) = \frac{0+b}{2} = \frac{b}{2}$$

Similarly for the geometric mean: $\mathrm{GM}(0,b) = \sqrt{0 \times b} = \sqrt{0} = 0$.

However, there is a situation with the harmonic mean. The formula for HM(*a*,*b*) involves $1/a$. When $a = 0$ we have a problem. Fortunately, the algebraic simplification displayed for the harmonic mean formula (see formula * on the previous page) computes for $a = 0$ and $b > 0$:

$$\mathrm{HM}(0,b) = \frac{2\times 0\times b}{0+b} = \frac{0}{b} = 0$$

Although the formulas for the arithmetic, geometric, and harmonic means can be expanded to accommodate $a = 0$, the strict mean inequality does not carry over for $a = 0$. For $a = 0$ we have

$$0 = \min(0,b) = \mathrm{HM}(0,b) = \mathrm{GM}(0,b) < \mathrm{AM}(0,b) = b/2 < \max(0,b) = b$$

However we can expand the scope of the inequality to cover $0 \le a < b$. In this event, we always have

$$a = \min(a,b) \le \mathrm{HM}(a,b) \le \mathrm{GM}(a,b) < \mathrm{AM}(a,b) < \max(a,b) = b$$

Finally we merely note that one can also extend calculating these means for more than two numbers. Suppose that $a_1, a_2, \dots, a_n$ are n positive numbers. Then,

$$\mathrm{AM}(a_1, a_2, \dots, a_n) = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

$$\mathrm{GM}(a_1, a_2, \dots, a_n) = \sqrt[n]{a_1 \times a_2 \times \cdots a_n}$$

$$\mathrm{HM}(a_1, a_2, \dots, a_n) = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$$

However, in this course we will not have occasion to apply these means beyond two numbers. We also caution that the alternative forms for HM(a,b) given in (*) do not apply beyond two numbers.

5. An Application: Rounding Decimal Numbers

Suppose that we have a positive decimal number that is not a whole number; i.e., a positive number with a genuine decimal part. Now, decimal fractions can be annoying. Hence, applying the classic research solution *denial* (what decimal fraction?) may provide convenience. Suppose that the integer part of our decimal number is n. We can then round the decimal down to n or round it up to n+1 to eliminate the decimal fraction. We will denote the rounding of a positive decimal number q by rnd(q). Note that rnd(q) will be either n or n+1.

There are five ways, depending on the application, to round a positive decimal number, q, with integer part n. In particular, rnd(q) =

1. $\min(n, n+1) = n$ — This is the "round down" option.
2. $\max(n, n+1) = n + 1$ — This is the "round up" option.
3. $n + 1 \iff q > n + ½ \iff q > \mathrm{AM}(n, n+1)$ — The usual method of rounding a decimal.
4. $n + 1 \iff q > \mathrm{GM}(n, n+1)$ — The geometric mean option.
5. $n + 1 \iff q > \mathrm{HM}(n, n+1)$ — The harmonic mean option.

To illustrate, let's work through an example. We apply the five methods to illustrate how to round 2.437. Should I round up to 3 or round down to 2?

1. rnd(2.437) = 2 — min option: round down
2. rnd(2.437) = 3 — max option: round up
3. rnd(2.437) = 2 — usual rounding method: $2.437 < \mathrm{AM}(2,3) = 2.5$
4. rnd(2.437) = 2 — geometric mean option: $2.437 < \mathrm{GM}(2,3) = 2.4494\cdots$
5. rnd(2.437) = 3 — harmonic mean option: $2.437 > \mathrm{HM}(2,3) = 2.4$

Exercises

1. Find the following averages.

    a. min(7,30)
    b. max(7,30)
    c. AM(7,30)
    d. GM(7,30)
    e. HM(7,30)

2. Compute the arithmetic, geometric, and harmonic mean for each pair of numbers.

    a. 3 and 48
    b. 4 and 7
    c. 8 and 20
    d. 8 and 9
    e. 6 and 12

3. Complete the table by rounding each of the given decimal numbers using the indicated method.

Decimal	Method				
	min	max	AM	GM	HM
3.2					
1.4					
1.463					
5.6					

4. **Sabermetrics** is the mathematical and statistical analysis of baseball records.[2] Most baseball fans are familiar with routine measures such as batting average, on base percentage, and even the power batting average where each hit is weighted by the number of bases reached by the hit. Some baseball aficionados find the **power-speed number** for a player a useful statistic to measure a player's clutch offensive productivity. The power-speed number is the harmonic mean of a player's home runs and stolen bases. The lifetime major league leaders for the power-speed index are the Giants legend Barry Bonds (613.90) and the *A's* Ricky Henderson.[3]

    a. Lifetime, Ricky Henderson hit 297 home runs and had 1406 stolen bases. What was Ricky Henderson's lifetime power-speed number?
    b. Ricky Henderson played 14 years for the Oakland *A's*. During this time with the *A's*, Henderson hit 167 home runs and stole 867 bases. What was Henderson's power-speed number at Oakland?

5. Compute the arithmetic, geometric and harmonic mean for each set of numbers.

    a. 1, 3, 5.
    b. 10, 12, 15, 18, 20.

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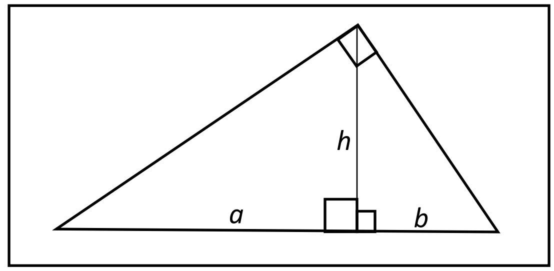
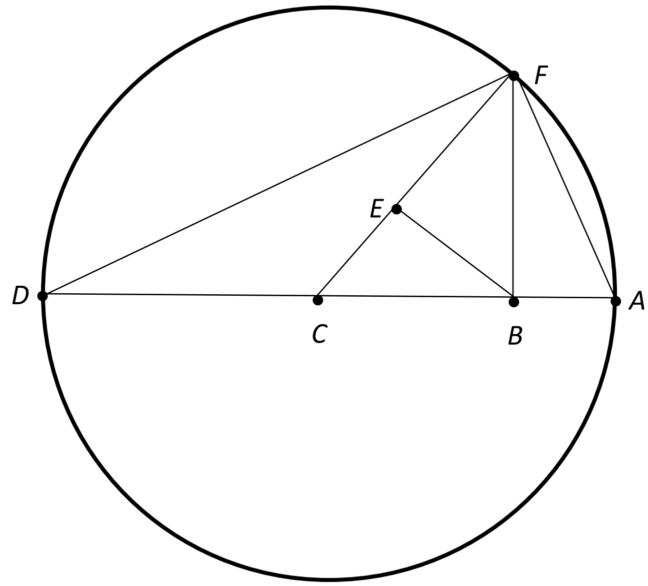
[2] http://www-math.bgsu.edu/~albert/papers/saber.html.
[3] http://www.baseball-reference.com/leaders/power_speed_number_career.shtml.

6. Use the adjacent figure to show that *h* is the geometric mean of *a* and *b*. The three indicated angles in the figure are all right angles. [Hint: The Pythagorean Theorem can come in handy.]

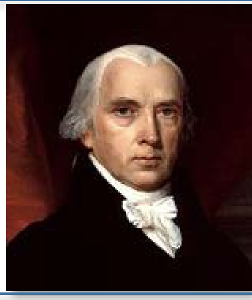


7. (Challenge) Consider the circle shown at right. Let $|\overline{AB}|$ = *a* and $|\overline{BD}|$ = *b*. The center of the circle is at *C*. $\overline{FB} \perp \overline{AD}$ and $\overline{BE} \perp \overline{CF}$.

    a. Verify that AM(*a*,*b*) = $|\overline{AC}|$.
    b. Verify that GM(*a*,*b*) = $|\overline{BF}|$.
    c. Verify that HM(*a*,*b*) = $|\overline{EF}|$.
    d. Explain why the diagram is a "proof without words" for the mean inequality.



Section 2: Basic Divisor Methods
The Congressional Apportionment Problem Based on the Census 1790-1840:



"… no political problem is less susceptible of a precise solution than that which relates to the number most convenient for a representative legislature, …"

James Madison[4]
The Federalist 55

The Congressional Apportionment Problem is an engaging application of mathematics to an ongoing problem in American history. The cast includes many well-known characters including George Washington, Thomas Jefferson, Alexander Hamilton, John Quincy Adams, Daniel Webster, James K. Polk, and Franklin Roosevelt. The problem is deceptively easy to state.

> **Congressional Apportionment Problem (CAP).** Determine the number of seats each state gets in the United States House of Representatives based on the decennial census.

Any doubts about the meaning of the problem are quickly resolved by considering the current situation as displayed in Figure 1.

[4] https://en.wikipedia.org/wiki/James_Madison#/media/File:James_Madison.jpg.
[2] http://www.census.gov/2010census/data/apportionment-data.php.

Figure 1. The congressional apportionment map based on the 2010 census.[5]

There are currently 50 states in the Union and 435 seats in the U. S. House of Representatives. The apportionment population for the nation was 309,183,463 based on the 2010 census. Thus, each congressperson represents about 711,000 people. The distribution of seats is made on the basis of population. California, the most populous state, has 53 seats, followed by Texas with 36. Seven states have the minimum representation of 1 seat each: Alaska, Montana, Wyoming, North Dakota, South Dakota, Vermont, and Delaware.

1. Constitutional Guidelines

The U. S. Constitution specifies the basis for representation immediately following the Preamble. Article I, Section 1, states that all law making powers are vested in Congress consisting of a Senate and a House. Representation in the Senate is based on geography: each state has two senators.

Guidelines for the House are minimal but substantial. Representation is based on population as determined by a decennial census. The Constitution also sets criteria for the minimum and maximum House size. Each state must have at least one representative. Further, the House size "shall not exceed one for every thirty Thousand." This means that in general a congressperson may represent 30000 or more people, but not less.

The initial congressional apportionment is specified in Article I, Section 2, and is known as the constitutional apportionment. It was based on the framers' estimates of the state populations in 1787. This constitutional apportionment would remain in effect until reapportionment based on the first census.

The first Congress admitted Vermont as the fourteenth State in the Union. It also authorized the first U. S. census which began in 1790. Accordingly, each year ending in a zero is a census year. Congress also passed an enabling act anticipating statehood for Kentucky in the near future. Hence, the first census involved fifteen states. The census report was submitted

The U.S. Constitution: Article I

Section 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of the State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall be Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative; and until such enumeration shall be made, the State of new Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode-Island and Providence Plantation one, Connecticut five, New-York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

to Congress by President George Washington on 28 October 1791. To understand where we are today, we go back to this first census.

2. Re-apportionment based on the 1790 Census

Congress received the census report on a Friday and went to work on apportionment the following Monday. The main view was that the Senate represented the States and that the House represented the People. Congressmen wanted maximum representation for the people. They began with the question, how many people should a congressman represent? Their answer: 30000. The answer is known as the ratio of representation (or more simply the ratio), the constituency, or the divisor. Accordingly they divided 30000 into the population of each state to determine how many representatives each state deserved. They took only the integer part of the answer. They felt that any fractional remainder was not enough to justify an additional representative. The results are displayed in Figure 2, House Bill main column.[6] It took the House just one month to finalize their bill which was then sent to the Senate for concurrence.

The Senate felt that 33000 was a better answer to the question, how many people should a congressman represent? They applied the same methodology as the House but used 33000 as the divisor. The results are displayed in Figure 2, Senate Bill main column. It took the Senate one month to finalize its bill.

However, neither chamber of Congress would accede to the other resulting in an impasse. To break the gridlock Congress needed to come up with some out-of-the-box thinking.

Census			House Bill		Senate Bill	
State		Population	*d* = 30000	Seats	*d* = 33000	Seats
CT	5	236841	7.90	7	7.18	7
DE	1	55540	1.85	1	1.68	1
GA	3	70835	2.36	2	2.15	2
KY		68705	2.29	2	2.08	2
MD	6	278514	9.28	9	8.44	8
MA	8	475327	15.84	15	14.40	14
NH	3	141822	4.73	4	4.30	4
NJ	4	179570	5.99	5	5.44	5
NY	6	331589	11.05	11	10.05	10
NC	5	353523	11.78	11	10.71	10
PA	8	432879	14.43	14	13.12	13
RI	1	68446	2.28	2	2.07	2
SC	5	206236	6.88	6	6.25	6
VT	2	85533	2.85	2	2.59	2
VA	10	630560	21.02	21	19.11	19
US	67	3615920	120.53	**112**	109.57	**105**

Figure 2. The 1790 Census data with the first House and Senate apportionment bills; *d* represents the constituency divisor.

A new approach was offered by Federalists in the House. They suggested starting with the House size rather than the constituency. Once the House size is known, then the thinking is obvious: if a state has 10% of the population, then it should have 10% of the seats in the House. Accordingly they presented the Rule of Three to calculate each state's quota:

$$\boldsymbol{quota} = (House\ size) \times \frac{state\ population}{national\ population}$$

The term "Rule of Three" highlights that to compute the quota one must use three things: the House size, the state population, and the national population.

[6] The census figures are taken from Michel Balinski and H. Peyton Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, 2nd, Washington, D.C.: Brookings Institution Press, 2001: 158. These are the final and corrected census figures, not the original data submitted by President Washington. Initially Congress had to deal with incomplete returns from South Carolina and some corrections made on the fly. However, the results presented here are consistent with the results historically obtained. For a detailed account of the history, see Charles Biles, *Congressional Apportionment Based on the Census 1790*, available as an open resource download from http://nia977.wix.com/drbcap.

Federalists in the House used this idea to test the result of the House and Senate bills. Note in particular that the House size was never used in making the House and Senate bills—the House size was merely a result of the apportionment methodology. But, once a bill is finalized, then the Rule of Three can be applied to the resulting House size.

Figure 3 displays the result of testing the House and Senate bills with the Rule of Three. The House bill created a House with 112 seats. The House Bill, Quota column, displays each state's fair share based on a House size of 112. The concept of quota contains an intrinsic rule of fairness known as the **quota rule**. For example, Connecticut's fair share of 112 seats is 7.336. However, fractional seats are impossible; hence, Connecticut's fair share is at least 7 but no more than 8. In general, the quota rule asserts that a state's fair share must be the quota rounded down or rounded up. The Rule of Three exposes a quota rule violation for Virginia. Virginia's fair share quota of 112 seats is 19.531, yet the House bill gives Virginia 21 seats. This problem became an eventual deal-breaker for the House bill.

Census		House Bill			Senate Bill		
State	**Population**	***d*=30000**	**Seats**	**Quota**	***d*=33000**	**Seats**	**Quota**
CT 5	236841	7.90	7	7.336	7.18	7	6.877
DE 1	55540	1.85	1	1.72	1.68	1	**1.613**
GA 3	70835	2.36	2	2.194	2.15	2	2.057
KY	68705	2.29	2	2.128	2.08	2	1.995
MD 6	278514	9.28	9	8.627	8.44	8	8.088
MA 8	475327	15.84	15	14.723	14.40	14	13.803
NH 3	141822	4.73	4	4.393	4.30	4	4.118
NJ 4	179570	5.99	5	5.562	5.44	5	5.214
NY 6	331589	11.05	11	10.271	10.05	10	9.629
NC 5	353523	11.78	11	10.95	10.71	10	10.266
PA 8	432879	14.43	14	13.408	13.12	13	12.57
RI 1	68446	2.28	2	2.12	2.07	2	1.988
SC 5	206236	6.88	6	6.388	6.25	6	5.989
VT 2	85533	2.85	2	2.649	2.59	2	2.484
VA 10	630560	21.02	21	**19.531**	19.11	19	**18.310**
US 67	3615920	120.53	**112**	112	109.57	**105**	105

Figure 3. Quota Rule analysis of the first House and Senate apportionment bills.

Interestingly the Senate bill has no quota rule violation. Quota rule violations are possible for any apportionment using a constituency approach. A quota rule violation does not have to occur using a given divisor, as the Senate bill verifies, but it may occur as the House bill verifies. Although free of any quota rule violation, the Senate bill has an annoying feature from the viewpoint of the quota. Virginia's fair share of 105 seats is 18.310, yet the Senate bill gives Virginia 19 seats. In contrast, Delaware's fair share of 105 seats is 1.613, yet the bill gives Delaware only 1 seat. Is it really fair that a state with a lower decimal quota is rounded up over a state with a higher decimal quota? This apparent favoritism is the result of the round down criterion that was applied to the quotient = (state population)/divisor. Both the House and Senate bills rounded all quotients by rounding down. Using the rounding down procedure on all decimal quotients may lead to a biased favoritism in the quota. Such resulting favoritism always favors a larger state over a smaller state.

With the discovery of these two flaws in the constituency approach to apportionment, Federalists felt that they had leverage to advocate their plan based on a House size approach. Federalists advanced the idea that initiating apportionment on the constituency question got things off on the wrong foot as evidenced by the results. Instead of asking, how many people should a congressman represent, we start with asking, what should be the size of the House?

To advance their plan, Federalists advocated maximum representation for the people. They also used the 30000 figure from the Constitution but began by dividing 30000 into the national population which yielded 120.53. Accordingly, following the constitutional constraint that the size of the House may not exceed one in thirty-thousand, the maximum allowable House size is 120. Federalists then asked, what is each state's fair share of 120? The results are shown in Figure 4.

The subsequent method is known as **Hamilton's method** in today's literature. The method first applies the Rule of Three to obtain each state's fair share quota for the given House size of 120 members. The method then allocates the lower quota (the quota rounded down) to each state. This distributes 111 of the 120 seats. There then remains 9 seats to distribute among the 15 states. These 9 seats are awarded to the 9 states with the largest decimal component in the quota. The decimal components of the quota may be thought of as a priority list. Accordingly, New Jersey with decimal component .96 is awarded the 112th seat and New Hampshire with decimal component .71 is awarded the 120th seat, completing the distribution of the 120 seats.

Census		Hamilton's Method			
State	**Population**	***h* = 120**	**Quota**	**Lower Q**	**Appt**
CT 5	236841		7.86	7	8
DE 1	55540		1.84	1	2
GA 3	70835		2.35	2	2
KY	68705		2.28	2	2
MD 6	278514		9.24	9	9
MA 8	475327		15.77	15	16
NH 3	141822		4.71	4	5
NJ 4	179570		5.96	5	6
NY 6	331589		11.00	11	11
NC 5	353523		11.73	11	12
PA 8	432879		14.37	14	14
RI 1	68446		2.27	2	2
SC 5	206236		6.84	6	7
VT 2	85533		2.84	2	3
VA 10	630560		20.93	20	21
US 67	3615920	120.531	120	111	120

Figure 4. The first apportionment bill passed by Congress.

The result had several remarkable advantages going for it. First, there are no quota rule violations; in fact, there can't be any quota rule violations since the distribution of seats is founded on the quota. Automatically, each state is given either the quota rounded down or rounded up as need be. Second, there cannot be any biased favoritism since additional seats are distributed according to largest fractions. Hence, the objections to the House and Senate bills were automatically overcome. Further, by a freak happenstance of the data, the seats allocated (Figure 4, Hamilton's Method, Appt column) correspond to an ordinary rounding of the quota. Even better, each state that was given an additional seat beyond the lower quota had a decimal fraction greater than .7 and each state given the lower quota had a decimal fraction less than .4. With all this going for it, this bill broke the House-Senate gridlock and became the first apportionment bill passed by Congress. On 26 March 1792, five months after receiving the census, Congress sent the bill to President Washington for his approval and signature.

President Washington vetoed the bill. The veto is significant for three reasons.

- It was the first presidential veto in U.S. history.
- It was the only veto of Washington's first administration.
- Washington justified his veto based on his interpretation of the Constitution.

The House size of 120 yields 3615920/120 = 30133 when applied to the U.S. population as a whole. But, when applied to Connecticut, 236841/8 = 29605. Washington insisted that the constitutional constraint that the size of the House shall "not exceed one for every thirty Thousand" must be satisfied by each state individually, not just the nation as a whole. After Washington's veto, Congress quickly passed the original Senate bill which Washington signed on 14 April 1792.

3. Basic Divisor Methods

The debate over re-apportionment based on the 1790 census displayed two approaches to the congressional apportionment problem: a constituency approach and a House size approach. A **constituency approach** is based on the question, how many people should a congressman represent? A **House size approach** is based on the question, how many seats should there be in the House?

The method used to construct the original Senate bill that eventually became the first apportionment act set precedent and was used for the next five censuses (see Figure 5). The method is called a **basic divisor method** and is based on a constituency approach. It involves a 3-step algorithm:

Step 1. Determine how many people a congressman should represent. Answer: d.

Step 2. Calculate each state's quotient:
quotient = (state population)/d.

Step 3. Round the quotient to obtain the state's apportionment.

1790: $s = 15$, $d = 33000 \Rightarrow h = 105$
1800: $s = 16$, $d = 33000 \Rightarrow h = 141$
1810: $s = 17$, $d = 35000 \Rightarrow h = 181$
1820: $s = 24$, $d = 40000 \Rightarrow h = 213$
1830: $s = 24$, $d = 47700 \Rightarrow h = 240$
1840: $s = 26$, $d = 70680 \Rightarrow h = 223$

Figure 5. A basic divisor method applied to the first six censuses; s represents the number of states, d the divisor, and h the House size.

The apportionment act based on the 1790 census used a basic divisor method in which each state's quotient was rounded down. We refer to this method as **Jefferson's method**, or more completely, Jefferson's basic divisor method. Jefferson's method was used for apportionments based on the census from 1790 to 1830, inclusive.

Flaws with Jefferson's method were evident from the start, but new quota rule violations demanded attention. Alternate proposals for rounding the decimal quotient surfaced. During the 1830 census-based apportionment debates, Daniel Webster, chair of the Senate apportionment committee, received letters from John Quincy Adams, a representative from Massachusetts, and James Dean, a mathematics professor at the University of Vermont. Thinking about alternatives proposed by Adams and Dean, Webster devised his own. Thus, four variations of the basic divisor method, all dealing with how to round a decimal (the quotient), were available to Webster.

- Jefferson: round down.
- Adams: round up.
- Dean: round down or up depending on which option gives a state's constituency closer to the divisor.
- Webster: round normally.

For apportionment based on the 1830 census Congress used the precedent Jefferson method. However, other methods were now on the table. Apportionment based on the 1840 census used a constituency approach with the divisor 70680 and, for the first time, Webster's method for rounding the quotient. This resulted in a House with 233 members. It was the only time in U.S. history that the House size decreased as a result of the decennial census-based reapportionment process.

4 Basic Divisor Methods

Step 3 in the basic divisor method involves rounding the quotient. During the debates based on the 1830 census, Daniel Webster had four proposals for how to round the quotient. Jefferson's, Adams's, and Webster's rounding criteria are easily familiar. We now take a closer look at Dean's method by looking at an example.

In 1830 the US population was 11,931,578.
Consider: constituency = 50,000 people.
Vermont's population: 280,657.
Vermont's quotient: 280,657/50,000 = 5.613.

At this point, Jefferson apportions 5 seats to Vermont; Adams, 6 seats.
With 5 seats the constituency is 280,657/5 = 56,131.
With 6 seats the constituency is 280,657/6 = 46,776.
A constituency of 46,776 is closer to the target constituency of 50,000; hence, Dean awards Vermont 6 seats.

Dean's and Webster's methods are similar in their thinking. Webster's method involves rounding the decimal quotient normally; i.e., if the decimal fraction is less than .5, then round down, otherwise round up. Denote the quotient by q and let n be the integer part of q. Then, q rounded down is n and q rounded up is $n+1$. Rounding q normally is equivalent to the criterion: round up if and only if q is greater than the arithmetic mean of the round down, round up options; i.e., $q > \text{AM}(n,n+1)$. Dean's method is mathematically equivalent to the criterion: round up if and only if q is greater than the harmonic mean of the round down, round up options; i.e., $q > \text{HM}(n,n+1)$.[7]

The four rounding options may be expressed in terms of the quotient as follows. A state's apportionment is obtained by rounding the quotient, q, where you round up if and only if

Jefferson:	$q > \max(n,n+1)$	Since this can't happen, always round down.
Adams:	$q > \min(n,n+1)$	Since this always happens, always round up.
Webster:	$q > \text{AM}(n,n+1)$	Round normally.
Dean:	$q > \text{HM}(n,n+1)$	Round by closest constituency.

For illustration, we now apply these four basic divisor methods to the 1810 census (see Figure 6). Although there is a lot of data in the displayed spreadsheet, one can quickly grasp the main elements. First, **Census 1810** lists the 17 states with their populations. Second, Congress used a constituency approach with a congressman representing **35000** people. Third, 35000 is divided into each state's population to determine each state's **Quotient**. Fourth, the quotient is rounded applying the four variations: **Jefferson**, **Adams**, **Webster**, and **Dean**. With the fixed constituency of 35000, the four rounding methods each lead to a different House size. Since Jefferson rounds all quotients down, this method produces the smallest House with 181 members. Since Adams rounds all quotients up, this method produces the largest House size with 198 members. Since Webster and Dean round some states up and other states down, they produce a House with an intermediate size. Note that Dean's method produces a House with one more member than Webster's method. The involved state is Connecticut whose quotient is

Census 1810		*d* = 35000				
State	Population	Quotient	Jefferson	Webster	Dean	Adams
CT	261818	7.4805	7	7	8	8
DE	71004	2.0287	2	2	2	3
GA	210346	6.0099	6	6	6	7
KY	374287	10.6939	10	11	11	11
MD	335946	9.5985	9	10	10	10
MA	700745	20.0213	20	20	20	21
NH	214460	6.1274	6	6	6	7
NJ	241222	6.8921	6	7	7	7
NY	953043	27.2298	27	27	27	28
NC	487971	13.9420	13	14	14	14
OH	230760	6.5931	6	7	7	7
PA	809773	23.1364	23	23	23	24
RI	76931	2.1980	2	2	2	3
SC	336569	9.6163	9	10	10	10
TN	243913	6.9689	6	7	7	7
VT	217895	6.2256	6	6	6	7
VA	817594	23.3598	23	23	23	24
US	6575234	188.1222	**181**	**188**	**189**	**198**

Figure 6. The 1910 census.

[7] For the mathematical derivation of these equivalencies, see Charles Biles, *Congressional Apportionment Based on the Census 1800-1840*: 50-53; available as an open-source download from http://www.nia977.wix.com/drbcap.

7.4805. Rounding normally, Webster rounds the quotient down and awards Connecticut 7 seats. However, Dean's rounding criterion is to round up if the quotient is larger than the harmonic mean of the round-down, round-up options. Here, HM(7,8) = 7.466···. Thus, Dean awards Connecticut 8 seats.

Hence, apportionment methodology matters. The criterion for rounding the quotient matters. Although different methods can produce the same result, they may all produce different results as exhibited by congressional apportionment based on the 1810 census.

Re-apportionment resulting from the first six censuses was accomplished using a basic divisor method. The first five re-apportionments used Jefferson's method. The sixth, based on the 1840 census, used Webster's method. However, the basic divisor method became subject to serious political manipulations and Congress looked for an alternative to eliminate gaming the system. Accordingly, in the apportionment debate based on the 1850 census Congress abandoned the basic divisor approach and applied a House size approach. This transformed the congressional apportionment problem into a mathematical apportionment problem.

Exercises

1. Consider the 1790 census.
   A. Create a spreadsheet showing the effects of applying the four basic divisor methods (Jefferson, Adams, Webster, Dean) to the 1790 census using the divisor 33000.
   B. Do the four methods lead to different results?

2. Repeat Exercise 1 for the 1800 census using the divisor 33000.

3. Repeat Exercise 1 for the 1820 census using the divisor 40000.

4. Repeat Exercise 1 for the 1830 census using the divisor 47700.

5. Repeat Exercise 1 for the 1840 census using the divisor 70680.

6. The state of Delaware has three counties: Kent, New Castle, and Sussex.[8]

   A. Complete the following table to apportion the Delaware House of Representatives using the indicated basic divisor method (see Figure 6 for illustration).

Census 2010		***d* = 20000**				
County	**Population**	**Quotient**	**Jefferson**	**Webster**	**Dean**	**Adams**
Kent	162310	8.1155				
New Castle	538479	26.92395				
Sussex	197145	9.85725				
Delaware	897934	44.8967				

[8] Exercises 6 and 7 are adapted from *Mathematics and Society: A Survey of mathematics for the liberal arts major*, 2012: 76-9. Available from http://www.opentextbookstore.com/mathinsociety/.

B. Repeat A. using a divisor of 21900.
C. Repeat A. using a divisor or 25000.

7. The state of Rhode Island has five counties: Bristol, Kent, Newport, Providence, and Washington. Complete the following table to apportion the Delaware House of Representatives using the indicated basic divisor method (see Figure 6 for illustration).

Census 2010		***d* = 14000**				
County	**Population**	**Quotient**	**Jefferson**	**Webster**	**Dean**	**Adams**
Bristol	49875	3.5625				
Kent	166158	11.8684				
Newport	82888	5.9206				
Providence	626667	44.7619				
Washington	126979	9.0699				
Rhode Island	1052567	75.1834				

Section 3: Quota Methods

Congressional apportionment based on the 1850 census saw a major evolution in methodology. Apportionment acts based on the census 1790-1840 were founded on the basic divisor method. In addition to problems caused by occasional quota rule violations, politicians became adept at gaming the system. In a single day during debates based on the 1840 census, 59 different proposals for a divisor were made in the House, including numbers such as 62279, 59241, and 53999. The House bill settled on 50179. On one day in the Senate there were 27 proposals for a divisor. The Senate settled on 70680. The apportionment act of 1842 finally specified 70680 and Webster's method.[9]

In 1850 Representative Samuel Vinton (Whig-Ohio) proposed a bill to stop the political gamesmanship. His plan established the House size in advance of the census and then set congressional apportionment on automatic pilot. The apportionment act of 1850 specified a House size of 233 with distribution calculated by the Census Office using Hamilton's method.

The Quota Method Algorithm

The apportionment act of 1850, also known as the Vinton act, transformed congressional apportionment into a modern day mathematical apportionment problem. In today's mathematics literature, apportionment is a fixed-resources problem. Formally, apportionment is the problem of distributing a fixed number of units among two or more groups.

Arguably the major problem in the world today is the equitable distribution of resources. Apportionment is thus applicable to a wide range of problems. There are several guidelines we must observe to create equitable solutions to apportionment problems.

Apportionment Guidelines

1. The units being distributed can exist only in whole numbers.
2. We must use all of the units being distributed and we cannot use any more.
3. Each group must get at least one of the units being distributed.
4. The number of units assigned to each group should be at least approximately proportional to the assets of the group. (Exact proportionality isn't possible because of the whole number requirement, but we should try to be close, and in any case, if Group A is larger than Group B, then Group B shouldn't get more units than Group A.)

These guidelines transform into a 4-step quota method algorithm.

Step 1. Determine the number of resource units to be distributed.

Step 2. Determine each group's **quota** by applying the rule of three:

$$Group\ Quota = (Resource\ Units) \times \frac{Group\ Assets}{Total\ Asssets}$$

Step 3. Allocate each group its lower quota (the quota rounded down).

[9] Charles Biles, *Congressional Apportionment Based on the Census 1850-1900*: 2-17. Available on-line at http://nia977.wix.com/drbcap.

Step 4. Distribute any remaining resource units according to a priority list based on each group's assets.

For congressional apportionment of the U.S. House of Representatives, the quota method algorithm uses a state's population as the measure of its assets since the Constitution mandates that apportionment shall be based on population.

Step 1. Select the House size, *h*. Currently, *h* = 435.
Step 2. Calculate each state's quota:

$$Quota = h \times \frac{state\ population}{national\ population}$$

Step 3. Allocate each state its lower quota.
Step 4. Distribute any remaining seats according to a priority list based on each state's population.

We examine three ways of completing Step 4: Hamilton, Lowndes, and Hill. Hamilton's method is what is usually applied and initially appeared during the apportionment debates based on the 1790 census. Lowndes's method was proposed during the debates based on the 1820 census.[10] Hill's method was proposed during the debates based on the 1910 census.[11]

Almost always Step 3 distributes most but not all of the House seats. Further, the number of remaining seats for distribution is less than the number of states. Hence, to complete Step 4 we need to create a priority list to determine which states get the remaining seats. The priority lists are calculated as follows.

Hamilton: the decimal fraction of the quota.
Lowndes: lower quota constituency = (state population)/(lower quota)
Hill: (state population)/$\sqrt{L \times U}$ where *L* is the lower quota and *U* is the upper quota.

Hamilton's Method

We first saw Hamilton's method in the congressional apportionment debates based on the census of 1790. The original congressional bill, which President George Washington vetoed, was based on House size 120 with apportionment based on Hamilton's method where additional seats were allocated by largest fractions. We now work through two examples to highlight the details.

Example 1

Delaware has three counties: Kent, New Castle, and Sussex. The Delaware House of Representatives has 41 members. We will apportion this representation along county lines (which is *not* required) by applying Hamilton's method. The populations of the counties are based on the 2010 census.

Step 1. Delaware's House size is 41.

[10] Charles Biles, *Congressional Apportionment Based on the Census 1800-1840*: 23-27. Available on-line at http://nia977.wix.com/drbcap.
[11] Charles Biles, *Congressional Apportionment Based on the Census 1900-1930*: 29-35. Available on-line at http://nia977.wix.com/drbcap.

Step 2. Determine each county's quota by applying the rule of three.

Census 2010		Hamilton's Method		
County	Population	Quota	Lower	Seats
Kent	162310	7.4111		
New Castle	538479	24.5872		
Sussex	197145	9.0017		
Delaware	897934	41.0000		

Step 3. Determine each county's lower quota (quota rounded down).

Census 2010		Hamilton's Method		
County	Population	Quota	Lower	Seats
Kent	162310	7.4111	7	
New Castle	538479	24.5872	24	
Sussex	197145	9.0017	9	
Delaware	897934	41.0000	40	

Step 4. Step 3 distributes 40 of the 41 seats. The remaining seat goes to the county with the largest decimal part, New Castle.

Census 2010		Hamilton's Method		
County	Population	Quota	Lower	Seats
Kent	162310	7.4111	7	7
New Castle	538479	24.5872	24	25
Sussex	197145	9.0017	9	9
Delaware	897934	41.0000	40	41

Example 2

We apply Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties based on the census of 2010.

Step 1: The House size is 75.
Step 2: Determine each county's quota by applying the rule of three.

Census 2010		Hamilton's Method		
County	Population	Quota	Lower	Seats
Bristol	49875	3.5538		
Kent	166158	11.8395		
Newport	82888	5.9061		
Providence	626667	44.6528		
Washington	126979	9.0478		
Rhode Island	1052567	75.0000		

Step 3: Determine each county's lower quota.

Census 2010		Hamilton's Method		
County	Population	Quota	Lower	Seats
Bristol	49875	3.5538	3	
Kent	166158	11.8395	11	
Newport	82888	5.9061	5	
Providence	626667	44.6528	44	
Washington	126979	9.0478	9	
Rhode Island	1052567	75.0000	72	

Step 4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which in decreasing order of priority are Newport, Kent, and Providence.

Census 2010		Hamilton's Method		
County	Population	Quota	Lower	Seats
Bristol	49875	3.5538	3	3
Kent	166158	11.8395	11	12
Newport	82888	5.9061	5	6
Providence	626667	44.6528	44	45
Washington	126979	9.0478	9	9
Rhode Island	1052567	75.0000	72	75

Note that even though Bristol County's decimal part is greater than .5, it isn't big enough to get an additional representative because three other counties have greater decimal parts.

Hamilton's method obeys an intuitive rule of fairness called the quota rule. In general the quota is a quantity that represents each group's fair share of the resource units based on its assets. Hence, each group deserves at least its lower quota, but no more than its upper quota. In terms of congressional apportionment, the quota rule states the following.

Quota Rule. The final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, this means that each state should be allocated either its lower quota or upper quota.

Lowndes's Method

Lowndes's method differs from Hamilton's method in Step 4 of the quota method algorithm. Instead of using the quota's largest fractions to determine priority for an additional seat, Lowndes uses the constituency obtained from the lower quota. To illustrate, we rework Hamilton's Example 1 and Example 2 with Lowndes's method. Since Steps 1-3 are the same for both methods, we only present Step 4.

Example 1: Delaware

Step 4: Calculate each county's constituency based on the lower quota; i.e., country constituency = (county population)/(lower quota).

Census 2010		Lowndes's Method			
County	Population	Quota	Lower	Constituency	Seats
Kent	162310	7.4111	7	23187	8
New Castle	538479	24.5872	24	22437	24
Sussex	197145	9.0017	9	21905	9
Delaware	897934	41.0000	40		41

In contrast to Hamilton, Lowndes gives the additional seat to Kent instead of New Castle. As a result, Kent's county constituency is reduced from 23187 to 20288.

Example 2: Rhode Island

Step 4: Calculate each county's constituency based on the lower quota; i.e., country constituency = (county population)/(lower quota).

Census 2010		Lowndes's Method			
County	Population	Quota	Lower	Constituency	Seats
Bristol	49875	3.5538	3	16625	4
Kent	166158	11.8395	11	15105	12
Newport	82888	5.9061	5	16578	6
Providence	626667	44.6528	44	14242	44
Washington	126979	9.0478	9	14109	9
Rhode Island	1052567	75.0000	72		75

Giving each county its lower quota distributes 72 of 75 seats. The remaining three seats are given to the three counties with the highest constituency based on the lower quota. These counties in descending order are Bristol, Newport, and Kent. Note that Lowndes gives a seat to Bristol at the expense of Providence in comparison with Hamilton.

Hill's Method

Joseph Hill was a statistician in the Census Bureau at the time of the 1910 census. Hill was concerned about the paradoxes produced by Hamilton's method. He thought that he could rescue Hamilton's method by a different method of allocating an additional seat in Step 4 of the quota method algorithm. Hill recommended that the priorities for Step 4 be computed by (state population)/$\sqrt{L \times U}$ where *L* is the lower quota and *U* is the upper quota. Recall that $\sqrt{L \times U}$ is the geometric mean of *L* and *U*. To illustrate, we rework Hamilton's Example 1 and Example 2 with Hill's method. Since Steps 1-3 are the same for both methods, we only present Step 4.

Example 1: Delaware

Step 4: Calculate each county's Hill priority: population/$\sqrt{L \times U}$.

Census 2010		Hill's Method			
County	Population	Quota	Lower	Priority	Seats
Kent	162310	7.4111	7	21689	7
New Castle	538479	24.5872	24	21983	25
Sussex	197145	9.0017	9	20780	9
Delaware	897934	41.0000	40		41

Note that in this example Hill agrees with Hamilton.

Example 2: Rhode Island

Step 4: Calculate each county's Hill priority: population/$\sqrt{L \times U}$.

Census 2010		Hill's Method			
County	Population	Quota	Lower	Priority	Seats
Bristol	49875	3.5538	3	14397	4
Kent	166158	11.8395	11	14462	12
Newport	82888	5.9061	5	15133	6
Providence	626667	44.6528	44	14083	44
Washington	126979	9.0478	9	13384	9
Rhode Island	1052567	75.0000	72		75

Giving each county its lower quota distributes 72 of 75 seats. The remaining three seats are given to the three counties with the highest Hill priority values. These counties, in descending order, are Bristol, Newport, and Kent. Note that Lowndes and Hill agree and, in contrast to Hamilton, give a seat to Bristol at the expense of Providence.

Paradoxes

A quota method initially seems quite fair since it is based on the quota seen as a group's "fair share" based on the group's assets. The eventual deal-breaker for the basic divisor method was the appearance of quota rule violations and the fact that it became subject to extensive political gamesmanship. The deal-breaker for quota methods turned out to be unexpected mathematical paradoxes. The three most prominent paradoxes are the Alabama paradox, the New States Paradox, and the Population paradox.

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama. Based on the 1880 census William Walker, Director of the Census, noted a strange anomaly: at House size 299 Alabama had 8 seats, but at House size 300 Alabama got only 7 seats. The question arose, how can an increase in resource units result in a decreased group share? This motivated Congress to pay more attention to the results of Hamilton's method.

The deal breaker for Hamilton's method came during debates based on the 1910 census. The Alabama paradox shined a spotlight on Colorado at House size 357. Colorado received two seats at House size 357 but three seats for all other House sizes between 350 and 400. The Alabama paradox also spotlighted West Virginia with 5 seats at House size 351, only four seats at House size 352 and 353, and back up to 5 seats at House size 354. The paradox was so vivid for Maine that several congressmen became convinced that the Hamilton method was flawed beyond acceptance. Maine received three seats for House sizes 350-382, 386, and 389-390 but four seats for House sizes 383-385, 387-388, and 391-400.

The **New States Paradox** was noticed when Oklahoma became the 46th state in 1907. Oklahoma had enough population to qualify for five representatives in Congress. These five representatives were assigned to Oklahoma in addition to the 386 already apportioned to the 45 existing states. But statisticians noted a strange phenomenon. When Hamilton's method is applied to the 1900 census with Oklahoma included, the resulting apportionment is the same except that Maine gains a representative at the expense of New York. Why should the addition of a new state affect two others?

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative *to Maine*. But things can even be worse. It is possible that a state may gain in population yet lose a seat to a state that drops in population.

Exercises

Apply the methods of Hamilton, Lowndes, and Hill, to apportion Exercises 1-8.

1. A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject.

   Math: 330 English: 265 Chemistry: 130 Biology: 70

2. Reapportion the previous problem if the college can hire 20 tutors.

3. The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information below.

Shift	Morning	Midday	Afternoon	Evening
Average number of customers	95	305	435	515

4. Reapportion the previous problem if the store has 25 salespeople.

5. Three people invest in a treasure dive, each investing the amount listed below. The dive results in 36 gold coins. Apportion those coins to the investors.

   Alice: $7,600 Ben: $5,900 Carlos: $1,400

6. Reapportion the previous problem if 37 gold coins are recovered.

7. A small country consists of five states, whose populations are listed below. If the legislature has 119 seats, apportion the seats.

   Allora: 810,000 Bella: 473,000 Cara: 292,000 Dulce: 594,000 Esotico: 211,000

8. A small country consists of six states, whose populations are listed below. If the legislature has 200 seats, apportion the seats.

   Acadie: 3,411 Beau: 2,421 Cognac: 11,586 Dordogne: 4,494 Été: 3,126 Fleur: 4,962

9. A small country consists of three colorful states with populations as follows: Azure, 6,000; Brown, 6,000; Crimson, 2,000.

   A. Apply Hamilton's method to apportion a 10-seat legislature.
   B. If the legislature grows to 11 seats, use Hamilton's method to apportion the seats.
   C. Which apportionment paradox does this illustrate?

10. A state with five counties has 50 seats in their legislature. Using Hamilton's method, apportion the seats based on the 2000 census, then again using the 2010 census. Which apportionment paradox does this illustrate?

County	**2000 Population**	**2010 Population**
Jefferson	60,000	60,000
Clay	31,200	31,200
Madison	69,200	72,400
Jackson	81,600	81,600
Franklin	118,000	118,400

11. A school district has two high schools: Lowell, serving 1715 students, and Fairview, serving 7364. The district could only afford to hire 13 guidance counselors.

    A. Determine how many counselors should be assigned to each school using Hamilton's method.
    B. The following year, the district expands to include a third school, serving 2989 students. By increasing the number of counselors proportionally, how many additional counselors should be hired for the new school?
    C. After hiring that many new counselors, the district recalculates the reapportion using Hamilton's method. Determine the outcome.
    D. Does this situation illustrate any apportionment issues?

12. A small country consists of four states whose populations are listed below. The legislature has 116 seats. Apportion the seats using Hamilton's method. Does this illustrate any apportionment issues?

Altura: 33,700 Baja: 559,500 Coast: 141,300 Desert: 89,100

Section 4: Modified Divisor Methods

Apportionment acts based on the census 1790-1840 were fabricated using a basic divisor method: Jefferson for 1790-1830 and Webster for 1840. Apportionment acts based on the census 1750-1900 were based on Hamilton's quota method. Congress abandoned the basic divisor method based on quota rule violations and excessive political gamesmanship. Congress wanted to abandon the quota method during the debates based on the 1900 census. However, there are only two approaches for the congressional apportionment problem: the constituency approach and the House size approach. The constituency approach is initiated by the question, how many people should a congressman represent? The House size approach is initiated by the question, how many seats should be in the House? The constituency approach naturally leads to a basic divisor method. The House size approach naturally leads to a quota method. What should Congress do now that both methods seemed unsatisfactory?

Congress searched for an improved method, one that would preserve the ease and fairness of the two known approaches but would suppress the resulting objections that could occur. Consequently they produced a hybrid method that incorporated both approaches.

Modified Divisor Methods

A modified divisor method starts with the House size approach by setting the House size. The House size is then set aside as a goal, the predetermined answer for a basic divisor method. A basic divisor method needs a divisor, the ratio of representation. The divisor process starts by calculating the **standard divisor** defined as the total population divided by h. In congressional apportionment of the U. S. House of Representatives the standard divisor represents the constituency of an average congressional district based on the given House size and the national population.

In general, the result is a 5-step modified divisor method algorithm.

Step 1. Determine the House size, h.
Step 2. Calculate the standard divisor: (total population)/h.
Step 3. Calculate each state's quotient using the given divisor.
Step 4. Round each state's quotient to obtain each state's apportionment.
Step 5. If the apportionments all sum to h, then DONE;
ELSE, modify the divisor and GO TO Step 3.

Note that the algorithm begins by setting the House size. Then, the House size is set aside as the answer for a basic divisor method. The standard divisor supplies us with a ratio of representation for a basic divisor method. Normally, using the standard divisor, the resulting House size does not match h. If the resulting House size is smaller than h, then the divisor was too big; hence, decrease the divisor and return to Step 3. If the resulting House size is larger than h, then the divisor was too small; hence, increase the divisor and return to Step 3.

A method of rounding needs to be adopted for Step 4. Thus far we have seen four rounding methods for congressional apportionment: Jefferson (round down), Adams (round up), Webster (round normally), and Dean (round by closest constituency). Accordingly we have Jefferson's, Adams's, Webster's, and Dean's modified divisor methods. Recall that Webster's method is mathematically equivalent to round the quotient up if and only if the quotient is greater than the arithmetic mean of the round down, round up options. Similarly, Dean's method is mathematically equivalent to round the quotient up if and only if the quotient is greater than the harmonic mean of the round down, round up options. With the methods of Jefferson and Webster, one must examine the results to make sure that the constitutional minimum is satisfied that each state receive at least one representative. If not, then assign the seat, modify (increase) the divisor, and go back to Step 3.

We caution that for enacting apportionment acts Jefferson's method historically was never used in the context of a modified divisor method but only in the context of a basic divisor method.

We illustrate the algorithm by reworking the Delaware example using Jefferson's method and the Rhode Island example using Webster's method. We will always round the standard divisor down for computational purposes within the modified divisor algorithm.

Example 1

We apply the Jefferson modified divisor method for Delaware.

Step 1. The House size is 41.
Step 2. The standard divisor for Delaware = 897934/41 = 21900.
Step 3. Calculate each state's quotient using the given divisor.
Step 4. Round each state's quotient to obtain each state's apportionment. For the Jefferson method all quotients are rounded down to obtain the number of seats.

Census 2010		Jefferson's Method	
County	**Population**	**Quotient**	**Seats**
Kent	162310	7.4114	7
New Castle	538479	24.5881	24
Sussex	197145	9.0021	9
Delaware	897934	41.0016	40

Step 5. If the apportionments all sum to *h*, then DONE;
ELSE, modify the divisor and GO TO Step 3.

Jefferson's method with divisor 21900 yields a House size of 40, one short of our goal of 41. Hence, since the resulting House size is too small, then our divisor, 21900 was too large. We modify the divisor down to 21000 and go back to Step 3.

Census 2010		Jefferson's Method	
County	**Population**	**Quotient**	**Seats**
Kent	162310	7.7290	7
New Castle	538479	25.6419	25
Sussex	197145	9.3879	9
Delaware	897934	42.7588	41

Voila! Success! This time, the divisor of 21000 produces the desired House size of 41 using Jefferson's method.

A key thing to note with the modified divisor method is that we don't have any left-over seats to distribute; hence, there is no need for a priority list. By modifying the divisor within the basic divisor method, all the seats are distributed. We merely need to modify the initializing standard divisor to obtain the desired House size goal.

Example 2

We apply the Webster modified divisor method for Rhode Island.

Step 1. The House size is 75.
Step 2. The standard divisor for Rhode Island = 1052567/75 = 14034.
Step 3. Calculate each state's quotient using the given divisor.
Step 4. Round each state's quotient to obtain each state's apportionment. For the Webster method all quotients are rounded normally.

Census 2010		Webster's Method	
County	**Population**	**Quotient**	**Seats**
Bristol	49875	3.5539	4
Kent	166158	11.8397	12
Newport	82888	5.9062	6
Providence	626667	44.6535	45
Washington	126979	9.0480	9
Rhode Island	1052567	75.0012	76

Step 5. If the apportionments all sum to *h*, then DONE;
ELSE, modify the divisor and GO TO Step 3.

Webster's method with divisor of 14034 yields a House size of 76, one more than our goal of 75. Hence, since the resulting House size is too large, then our divisor, 14034 was too small. We modify the divisor up to 15000 and go back to Step 3.

Census 2010		Webster's Method	
County	**Population**	**Quotient**	**Seats**
Bristol	49875	3.3250	3
Kent	166158	11.8397	12
Newport	82888	5.9062	6
Providence	626667	44.6535	45
Washington	126979	9.0480	9
Rhode Island	1052567	74.7723	75

Voila! Success! This time, the divisor of 15000 produces the desired House size of 75 using Webster's method.

Joseph Hill made his original proposal for adjusting the quota method from Hamilton's major fractions to priority numbers adjusted by a geometric mean. Shortly after Hill presented his idea, Edward Huntington, a mathematician who taught in the Engineering Department at Harvard, adapted Hill's idea to a modified divisor method by presenting another criterion for rounding a decimal. Suppose that q is a positive decimal number with a nonzero decimal fraction. Let n be q rounded down. Then q rounded up is $n+1$. Huntington's rounding criterion is round q up if and only if $q > \mathrm{GM}(n,n+1)$, the geometric mean of the round down, round up options. Accordingly a modified divisor method incorporating this method of rounding is called the Huntington-Hill method in today's apportionment literature. To illustrate we rework the Rhode Island Example using the Huntington-Hill method.

Example 3

We apply the Huntington-Hill (H-H) modified divisor method for Rhode Island.

Step 1. The House size is 75.
Step 2. The standard divisor for Rhode Island = 1052567/75 = 14034.
Step 3. Calculate each state's quotient using the given divisor.
Step 4. Round each state's quotient to obtain each state's apportionment. For the Webster method all quotients are rounded normally. The GM column is the geometric mean of the round down, round up options for the quotient. If Quotient > GM, then round up; otherwise, down. Note that the results are the same for Huntington-Hill and Webster at this point.

Census 2010		**H-H Method: 14034**		
County	**Population**	**Quotient**	**GM**	**Seats**
Bristol	49875	3.5539	3.4641	4
Kent	166158	11.8397	11.4891	12
Newport	82888	5.9062	5.4772	6
Providence	626667	44.6535	44.4972	45
Washington	126979	9.0480	9.4868	9
Rhode Island	1052567	75.0012		76

Step 5. If the apportionments all sum to *h*, then DONE;
ELSE, modify the divisor and GO TO Step 3.

The H-H method with divisor of 14034 yields a House size of 76, one more than our goal of 75. Hence, since the resulting House size is too large, then our divisor, 14034 was too small. We modify the divisor up to 15000 and go back to Step 3.

Census 2010		**H-H Method: 15000**		
County	**Population**	**Quotient**	**GM**	**Seats**
Bristol	49875	3.3250	3.4641	3
Kent	166158	11.8397	11.4891	12
Newport	82888	5.9062	5.4772	6
Providence	626667	44.6535	44.4972	45
Washington	126979	9.0480	9.4868	9
Rhode Island	1052567	74.7723		75

Voila! Success! This time, the divisor of 15000 produces the desired House size of 75 using the Huntington-Hill method.

In 1929 Congress enacted apportionment legislation that froze the House size at 435 for any apportionment based on the decennial census. House size 435 is still in use today. Further, in a 1941 amendment to the 1929 act, Congress permanently adopted the method of Huntington-Hill to apportion the U. S. House of Representatives. Computations are automatically done by the U. S. Census Bureau upon receipt of the census. Things will remain this way until Congress changes the law.

An obvious inquiry at this point is, how did Congress do at coming up with a workable methodology? Is there a best method for apportionment? The latter question was answered by two mathematicians, Michel Balinski and H. Peyton Young.

The Balinski and Young Impossibility Theorem

In 1982 Michel Balinski and H. Peyton Young proved that there is no perfect apportionment method.

The Balinski and Young Impossibility Theorem

There are no perfect apportionment methods. Any divisor method may violate the quota rule. Any method that avoids the Alabama paradox may produce quota rule violations. Further, any quota method is subject to paradoxes, especially the Alabama paradox.

The two main apportionment method flaws that Congress discovered in history were quota rule violations and the Alabama paradox. The Balinski-Young Theorem shows that it is impossible to avoid both. Any divisor method, whether basic or modified, may produce quota rule violations. The deal-breaker for the quota method was the Alabama paradox. However, the cost of avoiding this paradox is to use a method that may produce quota rule violations. To absolutely avoid quota rule violations, one must use a quota method, which may produce unwanted paradoxes.

The research of Payton and Young has shown that the Webster modified divisor method produces the best results as a general method. The Webster method of rounding is the only neutral quotient rounding mechanism. Adams, Dean, and Huntington-Hill all have an inborn bias that when it appears always favors small states over large states. Jefferson has an inborn bias that when it occurs always favors large states over small. Further, of all divisor methods Webster's is the least likely to violate the quota rule. It is not only least likely, but also unlikely. Webster can expect to produce a quota violation in the decennial reapportionment of the US House of Representatives once in about 16000 years.[12]

Exercises

1. Repeat the exercises in the previous section, *The Quota Method*, by applying the modified divisor methods of A. Jefferson, B. Webster, C. Dean, D. Huntington-Hill, E. Adams.

2. In any of the problems you worked in Exercise 1, were there any quota rule violations?

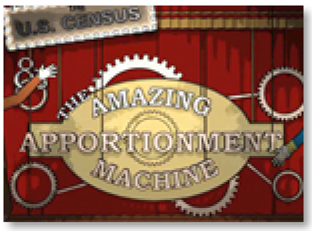
[12] Michel Balinski and H. Peyton Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, Brookings Institution Press, Washington, D. C., Second Edition, 2001: 81-2.

Section 5: It's a Matter of Priority

In this section we study the method currently used for computing congressional apportionment. Then we look at some proposals currently afloat for reforming the current situation.

Priority Calculation Techniques

In the previous section we studied the modified divisor method for apportionment. The method engaged a specific technique of computation in that apportionment was calculated for a specific House size. During debate based on the 1910 census, Walter Willcox introduced a serial method of computation that avoids the need to rerun the algorithm each time one wants to investigate a different House size.[13] The serial computation technique is one used today. We recommend that the reader pause to view the short (under two minutes) and charming video, *The Amazing Apportionment Machine*, available on the website of the U. S. Census Bureau.[14]



Willcox's technique first assigns one seat to each state. This immediately satisfies the Constitution's minimum requirement that each state shall have at least one seat. In today's House with 435 seats and 50 states, this initial step distributes 50 seats. Then there are 385 more seats to be distributed. Willcox asks, which state merits the 51st seat? 52nd seat? 53rd seat? Etc. To determine the serial merits of seats in the House, Willcox computed a set of priority numbers. The **priority number** (PN) for a state to receive an n+1st seat given that a state has n seats is defined by

$$\text{PN}(n) = \frac{\text{state population}}{\text{ave}(n,n+1)}$$

The various methods for computing an average lead to the various methods of apportionment.

Jefferson:	ave(n,n+1) = max(n,n+1)	
Adams:	ave(n,n+1) = min(n,n+1)	
Webster:	ave(n,n+1) = AM(n,n+1)	
H-H:	ave(n,n+1) = GM(n,n+1)	(H-H means Huntington-Hill)
Dean:	ave(n,n+1) = HM(n,n+1)	

For example, to compute the Webster priority number for a state to receive a second seat, divide a state's population by AM(1,2) = 1.5; for a state to receive a third seat (given that it already has two seats), divide the state's population by AM(2,3) = 2.5; etc. The averages needed to obtain the priority numbers for the different apportionment methods are:

Jefferson:	2, 3, 4, 5, 6, ⋯
Adams:	1, 2, 3, 4, 5, ⋯
Webster:	1.5, 2.5, 3.5, 4.5, 5.5, ⋯
H-H:	$\sqrt{2}, \sqrt{6}, \sqrt{12}, \sqrt{20}, \sqrt{30}, \cdots$
Dean:	4/3, 12/5, 24/7, 40/9, 60/11, ⋯

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[13] Charles Biles, *Congressional Apportionment Based on the Census 1900-1930*: 17-23. Available as an open-resource download from http://www.nia977.wix.com/drbcap.

[14] http://www.census.gov/2010census/mediacenter/census-data/census-apportionment-machine.php or http://www.census.gov/population/apportionment/.

We illustrate the process by serially apportioning the seats in the House based on the census of 1790 using Webster's method. Recall that the 1790 census involved 15 states. First, each state is initially given one seat. This distributes 15 seats. To compute priority numbers for each state, we divide each state's population by 1.5, 2.5, 3.5, 4.5, 5.5, etc. (see Figure 1).

Census 1790		Webster Priority Numbers				
State	**Population**	**1.5**	**2.5**	**3.5**	**4.5**	**5.5**
Connecticut	236841	157894	94736	67669	52631	43062
Delaware	55540	37027	22216	15869	12342	10098
Georgia	70835	47223	28334	20239	15741	12879
Kentucky	68705	45803	27482	19630	15268	12492
Maryland	278514	185676	111406	79575	61892	50639
Massachusetts	475327	316885	190131	135808	105628	86423
New Hampshire	141822	94548	56729	40521	31516	25786
New Jersey	179570	119713	71828	51306	39904	32649
New York	331589	221059	132636	94740	73686	60289
North Carolina	353523	235682	141409	101007	78561	64277
Pennsylvania	432879	288586	173152	123680	96195	78705
Rhode Island	68446	45631	27378	19556	15210	12445
South Carolina	206236	137491	82494	58925	45830	37497
Vermont	85533	57022	34213	24438	19007	15551
Virginia	630560	420373	252224	180160	140124	114647

Figure 1. Webster priority numbers for state to receive a 2nd, 3rd, 4th, 5th, and 6th seat.

The top priority to receive the next seat after the constitutional minimum will always be given to the largest state. In this situation based on the 1790 census Virginia has top priority to receive the 16th seat. Note that Virginia's priority number in the 1.5 column, 420373, is the largest priority number in the entire priority numbers matrix. After a priority number is used, it may not be used again. The next highest priority number, 316885, belongs to Massachusetts, which is given the 17th seat. The 18th seat goes to Pennsylvania with priority number 288586. The next priority number is 25224, which is Virginia's priority number to receive a third seat, so the 19th seat goes to Virginia. The 20th and 21st seats go to North Carolina (priority 235682) and New York (priority 221059). The 22nd seat goes to Massachusetts with priority number 190131. This priority approach generates the following sequence for seats 16-50 (after each state is initially given one seat each).

Seat 16: Virginia (2 seats)	Priority: 420373
Seat 17: Massachusetts (2 seats)	Priority: 316885
Seat 18: Pennsylvania (2 Seats)	Priority: 288586
Seat 19: Virginia (3 seats)	Priority: 252224
Seat 20: North Carolina (2 seats)	Priority: 235682
Seat 21: New York (2 seats)	Priority: 221059
Seat 22: Massachusetts (3 seats)	Priority: 190131
Seat 23: Maryland (2 seats)	Priority: 185676
Seat 24: Virginia (4 seats)	Priority: 180160
Seat 25: Pennsylvania (3 seats)	Priority: 173152
Seat 26: Connecticut (2 seats)	Priority: 157894
Seat 27: North Carolina (3 seats)	Priority: 141409

Seat 28: Virginia (5 seats)	Priority: 140124
Seat 29: South Carolina (2 seats)	Priority: 137491
Seat 30: Massachusetts (4 seats)	Priority: 135808
Seat 31: New York (3 seats)	Priority: 132636
Seat 32: Pennsylvania (4 seats)	Priority: 123680
Seat 33: New Jersey (2 seats)	Priority: 119713
Seat 34: Virginia (6 seats)	Priority: 114657

In order to proceed with the series, we need to expand the list of priority numbers. At this point we do not know Virginia's priority for a 7th seat. Virginia's priority for a 7th seat is 630560/6.5 = 97009. Accordingly, we continue as follows.

Seat 35: Maryland (3 seats)	Priority: 111406
Seat 36: Massachusetts (5 seats)	Priority: 105628
Seat 37: North Carolina (4 seats)	Priority: 101007
Seat 38: Virginia (7 seats)	Priority: 97009

To continue we need Virginia's priority for an 8th seat: 630560/7.5 = 84075.

Seat 39: Pennsylvania (5 seats)	Priority: 96195	
Seat 40: New York (4 seats)	Priority: 94740	
Seat 41: Connecticut (3 seats)	Priority: 94736	
Seat 42: New Hampshire (2 seats)	Priority: 94548	
Seat 43: Massachusetts (6 seats)	Priority: 86423	Next: PN(7) = 73127
Seat 44: Virginia (8 seats)	Priority: 84075	Next: PN(9) = 74184
Seat 45: South Carolina (3 seats)	Priority: 82494	
Seat 45: Maryland (4 seats)	Priority: 79595	
Seat 46: Pennsylvania (6 seats)	Priority: 78705	Next: PN(7) = 66579
Seat 47: North Carolina (5 seats)	Priority: 78561	
Seat 48: Virginia (9 seats)	Priority: 74184	Next: PN(9) = 74184
Seat 49: New York (5 seats)	Priority: 73686	
Seat 50: Massachusetts (7 seats)	Priority: 73127	Next: PN(7) = 63377

This serial computational technique can allow some interesting comparisons between methods. For example, Webster's method gives the 22nd seat to Massachusetts and the 23rd to Maryland. By comparison, Huntington-Hill reverses this order giving the 22nd seat to Maryland and the 23rd to Massachusetts. Further, Webster gives the 27th seat to North Carolina, 28th to Virginia, and the 29th to South Carolina. In contrast, Huntington-Hill gives the 27th seat to South Carolina, 28th to North Carolina, and the 29th to Virginia. Hence, the averaging method used to calculate priority numbers has consequences when the method is applied to distributing resources one at a time by priority.

The method in use today became official in congressional apportionment based on the 1940 census and uses the Huntington-Hill mechanism for creating priority numbers. The mechanism was originated by Joseph Hill during deliberations on apportionment based on the 1910 census. Hill's intuition was that the paradoxes manifested by Hamilton's method were the result of the mechanism of largest fractions. Hill developed his priority mechanism to avert paradoxes within Hamilton's method. Let's consider how this works by taking another look at apportionment based on the 1790 census (see Figure 2).

Census 1790		Hamilton' Method: *h* = 105			Hill: *h* = 105	
State	**Population**	**Quota**	**Lower**	**Seats**	**Priority**	**Seats**
Connecticut	236841	6.8774	6	7	36545	7
Delaware	55540	1.6128	1	2	39273	2
Georgia	70835	2.0569	2	2	28918	2
Kentucky	68705	1.9951	1	2	48582	2
Maryland	278514	8.0876	8	8	32823	8
Massachusetts	475327	13.8027	13	14	35234	14
New Hampshire	141822	4.1183	4	4	31712	4
New Jersey	179570	5.2144	5	5	32785	5
New York	331589	9.6288	9	10	34953	10
North Carolina	353523	10.2657	10	10	33707	10
Pennsylvania	432879	12.5700	12	13	34658	12
Rhode Island	68446	1.9876	1	2	48399	2
South Carolina	206236	5.9887	5	6	37653	6
Vermont	85533	2.4837	2	2	34919	3
Virginia	630560	18.3104	18	18	34097	18
US	3615920	105.0000	97	105		105

Figure 2. Apportionment based on the 1790 census for Hamilton's quota method with House size 105 using largest fractions. Hill's method uses priority numbers based on the geometric mean.

First, consider Hamilton's method with House size 105. Giving each state its lower quota distributes 97 of the 105 seats. The remaining 8 seats are given to the states with the 8 largest fractions to bring the total to 105. The Hill priority numbers are calculated by (state population)/squareroot(lower quota × upper quota). The remaining 8 seats are given to the states with the 8 largest Hill priority numbers. Note that in comparison to Hamilton's method the Hill method awards an additional seat to Vermont at the expense of Pennsylvania

We recommend that the reader visit the website of the Census Bureau for information regarding apportionment based on the 2010 census. In particular, the Census Bureau provides a serial apportionment for House sizes 51-440 based on the Huntington-Hill priority numbers.[15] If you'd like to do your own apportionment analysis based on the 2010 census you can download an Excel spreadsheet of the 2010 census.[16]

Future Considerations

We will consider four proposals for refining or reforming the congressional apportionment procedure: thirty-thousand.org, The Wyoming Rule, Neubauer and Carr, and H. Peyton Young. Three of the proposals, thirty-thousand.org, The Wyoming Rule, and Young's are documented on internet websites.

We strongly recommend visiting the website http://www.thirty-thousand.org. The group presents perhaps the most cogent arguments for reform, even though at first their proposed solution may seem silly. If one literally uses the name with the 2010 census, then one is applying a Jefferson basic divisor method with divisor 30000. This results in a House of size 10283. We can only leave it to your imagination how such a House would function, both politically and mechanically. The California delegation alone would have 1244 members, almost tripling the size of the current House.

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[15] https://www.census.gov/population/apportionment/files/Priority%20Values%202010.pdf.
[16] https://www.census.gov/population/apportionment/data/2010_apportionment_results.html.

Another interesting proposal is The Wyoming Rule (simply Google the term). The Wyoming Rule proposes applying a basic divisor method where the divisor is the population of the least populous state. From both the 2000 and 2010 census the least populous state is Wyoming; hence, the name. The rationale for The Wyoming Rule is simple. Currently California's population is 66 times that of Wyoming. Yet, California has 53 representatives and Wyoming has 1. Does this really satisfy the constitutional mandate that representatives shall be apportioned among the several states "according to their respective numbers?" The Wyoming Rule would correct this discrepancy.

If The Wyoming Rule were applied to the 2000 census, then the resulting House would have been 568 using the Huntington-Hill method. In 2010 the resulting House size would have been 542 using Huntington-Hill. In contrast, using the 2010 census, Dean's method yields 543 seats (additional seat to Hawaii) and Webster's method yields 540 seats (New Jersey and South Dakota lose a seat). Hence, with The Wyoming Rule the House size fluctuates from decade to decade.

A third interesting proposal was given by Neubauer and Gartner. They argued that House size and apportionment method should be considered together. They defined a House size is **agreeable** means that the methods of Hamilton, Dean, Huntington-Hill, and Webster all agree (produce the same apportionment results). Their proposal was that "after each census, increase the House size to the first agreeable House size larger than the existing House size." [17] The current House size 435 was not agreeable based on the 2000 census. The first agreeable House size was 477. The House size 435 was not agreeable based on the 2010 census. The first agreeable House size is 871, leading to a dramatic increase in House size.

The fourth and simplest proposal was made by H. Peyton Young in a paper prepared for the U. S. Census Bureau.[18] The proposal is merely to replace the current Huntington-Hill method of rounding the quotient with Webster's method. Recent mathematical research has shown that Webster's method is the only rounding method free of large-state/small-state bias. Based on the 2010 census, Webster would have given North Carolina an additional seat at the expense of Rhode Island.

Exercises

1. Apply the priority number matrix given in Figure 1 to continue the sequence for seats 51-60.

2. Select any apportionment problem thus far presented in the course or select one from an internet search. Initialize by giving one resource unit to each participant. Construct a priority number matrix (see Figure 1) for the first five priority levels for each participant using the following methods.

   A. Jefferson B. Adams C. Webster D. Huntington-Hill E. Dean

3. Apply the priority numbers matrix from Exercise 2 to construct a serial apportionment list for the distribution of the next 10 resource items.

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[17] Michael Neubauer and Margo Gartner, *A Proposal for Apportioning the House*, Political Science and Politics 44(1), January 2011: 77-79.
[18] H. Peyton Young, *Fairness in Apportionment*, January, 2004. Available as download from https://www.census.gov/history/pdf/Fairness_in_Apportionment_Young.pdf.