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The Congressional Apportionment Problem Based on the Census 1790-1840: Basic Divisor Methods



“... no political problem is less susceptible of a precise solution than that which relates to the number most convenient for a representative legislature, ...”

James Madison¹
The Federalist 55

The Congressional Apportionment Problem is an engaging application of mathematics to an ongoing problem in American history. The cast includes many well-known characters including George Washington, Thomas Jefferson, Alexander Hamilton, John Quincy Adams, Daniel Webster, James K. Polk, and Franklin Roosevelt. The problem is deceptively easy to state.

Congressional Apportionment Problem (CAP). Determine the number of seats each state gets in the United States House of Representatives based on the decennial census.

Any doubts about the meaning of the problem are quickly resolved by considering the current situation as displayed in Figure 1.



Figure 1. The congressional apportionment map based on the 2010 census.²

¹ https://en.wikipedia.org/wiki/James_Madison#/media/File:James_Madison.jpg.

² <http://www.census.gov/2010census/data/apportionment-data.php>.

There are currently 50 states in the Union and 435 seats in the U. S. House of Representatives. The apportionment population for the nation was 309,183,463 based on the 2010 census. Thus, each congressperson represents about 711,000 people. The distribution of seats is made on the basis of population. California, the most populous state, has 53 seats, followed by Texas with 36. Seven states have the minimum representation of 1 seat each: Alaska, Montana, Wyoming, North Dakota, South Dakota, Vermont, and Delaware.

1. Constitutional Guidelines

The U. S. Constitution specifies the basis for representation immediately following the Preamble. Article I, Section 1, states that all law making powers are vested in Congress consisting of a Senate and a House. Representation in the Senate is based on geography: each state has two senators.

Guidelines for the House are minimal but substantial. Representation is based on population as determined by a decennial census. The Constitution also sets criteria for the minimum and maximum House size. Each state must have at least one representative. Further, the House size “shall not exceed one for every thirty Thousand.” This means that in general a congressperson may represent 30000 or more people, but not less.

The initial congressional apportionment is specified in Article I, Section 2, and is known as the constitutional apportionment. It was based on the framers’ estimates of the state populations in 1787. This constitutional apportionment would remain in effect until reapportionment based on the first census.

The first Congress admitted Vermont as the fourteenth State in the Union. It also authorized the first U. S. census which began in 1790. Accordingly, each year ending in a zero is a census year. Congress also passed an enabling act anticipating statehood for Kentucky in the near future. Hence, the first census involved fifteen states. The census report was submitted to Congress by President George Washington on 28 October 1791. To understand where we are today, we go back to this first census.

The U.S. Constitution: Article I

Section 1. All legislative Powers herein granted shall be vested in a Congress of the United States, which shall consist of a Senate and House of Representatives.

Section 2. The House of Representatives shall be composed of Members chosen every second Year by the People of the several States, and the Electors in each State shall have the Qualifications requisite for Electors of the most numerous Branch of the State Legislature.

No Person shall be a Representative who shall not have attained to the Age of twenty five Years, and been seven Years a Citizen of the United States, and who shall not, when elected, be an Inhabitant of the State in which he shall be chosen.

Representatives and direct Taxes shall be apportioned among the several States which may be included within this Union, according to their respective Numbers, which shall be determined by adding to the whole Number of free Persons, including those bound to Service for a Term of Years, and excluding Indians not taxed, three fifths of all other Persons. The actual Enumeration shall be made within three Years after the first Meeting of the Congress of the United States, and within every subsequent Term of ten Years, in such Manner as they shall be Law direct. The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at Least one Representative; and until such enumeration shall be made, the State of new Hampshire shall be entitled to chuse three, Massachusetts eight, Rhode-Island and Providence Plantation one, Connecticut five, New-York six, New Jersey four, Pennsylvania eight, Delaware one, Maryland six, Virginia ten, North Carolina five, South Carolina five, and Georgia three.

2. Re-apportionment based on the 1790 Census

Congress received the census report on a Friday and went to work on apportionment the following Monday. The main view was that the Senate represented the States and the House represented the People. Congressmen wanted maximum representation for the people. They began with the question, how many people should a congressman represent? Their answer: 30000. The answer is known as the ratio of representation (or more simply the ratio), the constituency, or the divisor. Accordingly they divided 30000 into the population of each state to determine how many representatives each state deserved. They took only the integer part of the answer. They felt that the fractional remainder did not justify an additional representative. The results are displayed in Figure 2, House Bill main column.³ It took the House just one month to finalize their bill which was then sent to the Senate for concurrence.

The Senate felt 33000 was a better answer to the question, how many people should a congressman represent? Otherwise they applied the same methodology as the House. The results are displayed in Figure 2, Senate Bill main column. It took the Senate one month to finalize its bill.

However, neither chamber of Congress would accede to the other resulting in an impasse. To break the gridlock Congress needed to come up with some out-of-the-box thinking.

A new approach was offered by Federalists in the House. They suggested starting with the House size rather than the constituency. Once the House size is known, then the thinking is obvious: if a state has 10% of the population, then it should have 10% of the seats in the House. Accordingly they offered the Rule of Three to calculate each state's quota:

$$\text{quota} = (\text{House size}) \times \frac{\text{state population}}{\text{national population}}$$

The term "Rule of Three" highlights that to compute the quota one must use three things: the House size, the state population, and the national population.

Federalists in the House used this idea to test the result of the House and Senate bills. Note in particular that the House size was never used in making the House and Senate bills—the House size was merely a result of the apportionment methodology. But, once a bill is finalized, then the Rule of Three can be applied to the resulting House size.

Census		House Bill		Senate Bill	
State	Population	d = 30000	Seats	d = 33000	Seats
CT	5 236841	7.90	7	7.18	7
DE	1 55540	1.85	1	1.68	1
GA	3 70835	2.36	2	2.15	2
KY	68705	2.29	2	2.08	2
MD	6 278514	9.28	9	8.44	8
MA	8 475327	15.84	15	14.40	14
NH	3 141822	4.73	4	4.30	4
NJ	4 179570	5.99	5	5.44	5
NY	6 331589	11.05	11	10.05	10
NC	5 353523	11.78	11	10.71	10
PA	8 432879	14.43	14	13.12	13
RI	1 68446	2.28	2	2.07	2
SC	5 206236	6.88	6	6.25	6
VT	2 85533	2.85	2	2.59	2
VA	10 630560	21.02	21	19.11	19
US	67 3615920	120.53	112	109.57	105

Figure 2. The 1790 Census data with the first House and Senate apportionment bills; *d* represents the constituency divisor.

³ The census figures are taken from Michel Balinski and H. Peyton Young, *Fair Representation: Meeting the Ideal of One Man, One Vote*, 2nd, Washington, D.C.: Brookings Institution Press, 2001: 158. These are the final and corrected census figures, not the original data submitted by President Washington. Initially Congress had to deal with incomplete returns from South Carolina and some corrections made on the fly. However, the results presented here are consistent with the results historically obtained. For a detailed account of the history, see Charles Biles, *Congressional Apportionment Based on the Census 1790*, available as an open resource download from <http://nia977.wix.com/drbcap>.

Figure 3 displays the result of testing the House and Senate bills with the Rule of Three. The House bill created a House with 112 seats. The House Bill, Quota column, displays each state's fair share based on a House size of 112. The concept of quota contains an intrinsic rule of fairness known as the **quota rule**. For example, Connecticut's fair share of 112 seats is 7.336. However, fractional seats are impossible; hence, Connecticut's fair share is at least 7 but no more than 8. In general, the quota rule asserts that a state's fair share must be the quota rounded down or rounded up. The Rule of Three exposes a quota rule violation for Virginia. Virginia's fair share quota of 112 seats is 19.531, yet the House bill gives Virginia 21 seats. This problem became an eventual deal-breaker for the House bill.

Census		House Bill			Senate Bill			
State	Population	d=30000	Seats	Quota	d=33000	Seats	Quota	
CT	5	236841	7.90	7	7.336	7.18	7	6.877
DE	1	55540	1.85	1	1.72	1.68	1	1.613
GA	3	70835	2.36	2	2.194	2.15	2	2.057
KY		68705	2.29	2	2.128	2.08	2	1.995
MD	6	278514	9.28	9	8.627	8.44	8	8.088
MA	8	475327	15.84	15	14.723	14.40	14	13.803
NH	3	141822	4.73	4	4.393	4.30	4	4.118
NJ	4	179570	5.99	5	5.562	5.44	5	5.214
NY	6	331589	11.05	11	10.271	10.05	10	9.629
NC	5	353523	11.78	11	10.95	10.71	10	10.266
PA	8	432879	14.43	14	13.408	13.12	13	12.57
RI	1	68446	2.28	2	2.12	2.07	2	1.988
SC	5	206236	6.88	6	6.388	6.25	6	5.989
VT	2	85533	2.85	2	2.649	2.59	2	2.484
VA	10	630560	21.02	21	19.531	19.11	19	18.310
US	67	3615920	120.53	112	112	109.57	105	105

Figure 3. Quota Rule analysis of the first House and Senate apportionment bills.

Interestingly the Senate bill has no quota rule violation. Quota rule violations are possible for any apportionment using a constituency approach. A quota rule violation does not have to occur using a given divisor, as the Senate bill verifies, but it may occur as the House bill verifies. Although free of any quota rule violation, the Senate bill has an annoying feature from the viewpoint of the quota. Virginia's fair share of 105 seats is 18.310, yet the Senate bill gives Virginia 19 seats. In contrast, Delaware's fair share of 105 seats is 1.613, yet the bill gives Delaware only 1 seat. Is it really fair that a state with a lower decimal quota is rounded up over a state with a higher decimal quota? This apparent favoritism is the result of the round down criterion that was applied to the quotient = (state population)/divisor. Both the House and Senate bills rounded all quotients by rounding down. Using the rounding down procedure on all decimal quotients may lead to a biased favoritism in the quota. Such resulting favoritism always favors a larger state over a smaller state.

With the discovery of these two flaws in the constituency approach to apportionment, Federalists felt that they had leverage to advocate their plan based on a House size approach. Federalists advanced the idea that initiating apportionment on the constituency question got things off on the wrong foot as evidenced by the results. Instead of asking, how many people should a congressman represent, we start with asking, what should be the size of the House?

To advance their plan, Federalists advocated maximum representation for the people. They also used the 30000 figure from the Constitution but began by dividing 30000 into the national population which yielded 120.53. Accordingly, following the constitutional constraint that the size of the House may not exceed one in thirty-thousand, the maximum allowable House size is 120. Federalists then asked, what is each state's fair share of 120? The results are shown in Figure 4.

The subsequent method is known as **Hamilton's method** in today's literature. The method first applies the Rule of Three to obtain each state's fair share quota for the given House size of 120 members. The method then allocates the lower quota (the quota rounded down) to each state. This distributes 111 of the 120 seats. There then remains 9 seats to distribute among the 15 states. These 9

seats are awarded to the 9 states with the largest decimal component in the quota. The decimal components of the quota may be thought of as a priority list. Accordingly, New Jersey with decimal component .96 is awarded the 112th seat and New Hampshire with decimal component .71 is awarded the 120th seat, completing the distribution of the 120 seats.

The result had several remarkable advantages going for it. First, there are no quota rule violations; in fact, there can't be any quota rule violations since the distribution of seats is founded on the quota. Automatically, each state is given either the quota rounded down or rounded up as need be. Second, there cannot be any biased favoritism since additional seats are distributed according to largest fractions. Hence, the objections to the House and Senate bills were automatically overcome. Further, by a freak happenstance of the data, the seats allocated (Figure 4, Hamilton's Method, Appt column) correspond to an ordinary rounding of the quota. Even better, each state that was given an additional seat beyond the lower quota had a decimal fraction greater than .7 and each state given the lower quota had a decimal fraction less than .4. With all this going for it, this bill broke the House-Senate gridlock and became the first apportionment bill passed by Congress. On 26 March 1792, five months after receiving the census, Congress sent the bill to President Washington for his approval and signature.

Census		Hamilton's Method				
State	Population	$h = 120$	Quota	Lower Q	Appt	
CT	5	236841	7.86	7	8	
DE	1	55540	1.84	1	2	
GA	3	70835	2.35	2	2	
KY		68705	2.28	2	2	
MD	6	278514	9.24	9	9	
MA	8	475327	15.77	15	16	
NH	3	141822	4.71	4	5	
NJ	4	179570	5.96	5	6	
NY	6	331589	11.00	11	11	
NC	5	353523	11.73	11	12	
PA	8	432879	14.37	14	14	
RI	1	68446	2.27	2	2	
SC	5	206236	6.84	6	7	
VT	2	85533	2.84	2	3	
VA	10	630560	20.93	20	21	
US	67	3615920	120.531	120	111	120

Figure 4. The first apportionment bill passed by Congress.

President Washington vetoed the bill. The veto is significant for three reasons.

- It was the first presidential veto in U.S. history.
- It was the only veto of Washington's first administration.
- Washington justified his veto based on his interpretation of the Constitution.

The House size of 120 yields $3615920/120 = 30133$ when applied to the U.S. population as a whole. But, when applied to Connecticut, $236841/8 = 29605$. Washington insisted that the constitutional constraint that the size of the House shall "not exceed one for every thirty Thousand" must be satisfied by each state individually, not just the nation as a whole. After Washington's veto, Congress quickly passed the original Senate bill which Washington signed on 14 April 1792.

3. Basic Divisor Methods

The debate over re-apportionment based on the 1790 census displayed two approaches to the congressional apportionment problem: a constituency approach and a House size approach. A **constituency approach** is based on the question, how many people should a congressman represent? A **House size approach** is based on the question, how many seats should there be in the House?

The method used to construct the original Senate bill that eventually became the first apportionment act set precedent and was used for the next five censuses (see Figure 5). The method is called a **basic divisor method** and is based on a constituency approach. It involves a 3-step algorithm:

- Step 1. Determine how many people a congressman should represent. Answer: d .
- Step 2. Calculate each state's quotient:
quotient = (state population)/ d .
- Step 3. Round the quotient to obtain the state's apportionment.

1790: $s = 15, d = 33000 \Rightarrow h = 105$
1800: $s = 16, d = 33000 \Rightarrow h = 141$
1810: $s = 17, d = 35000 \Rightarrow h = 181$
1820: $s = 24, d = 40000 \Rightarrow h = 213$
1830: $s = 24, d = 47700 \Rightarrow h = 240$
1840: $s = 26, d = 70680 \Rightarrow h = 223$

Figure 5. A basic divisor method applied to the first six censuses; s represents the number of states, d the divisor, and h the House size.

The apportionment act based on the 1790 census used a basic divisor method in which each state's quotient was rounded down. We refer to this method as **Jefferson's method**, or more completely, Jefferson's basic divisor method. Jefferson's method was used for apportionment acts based on the census from 1790 to 1830, inclusive.

Flaws with Jefferson's method were evident from the start, but new quota rule violations demanded attention. Alternate proposals for rounding the decimal quotient surfaced. During the 1830 census-based apportionment debates, Daniel Webster, chair of the Senate apportionment committee, received letters from John Quincy Adams, a representative from Massachusetts, and James Dean, a mathematics professor at the University of Vermont. Thinking about alternatives proposed by Adams and Dean, Webster devised his own. Thus, four variations of the basic divisor method, all dealing with how to round a decimal (the quotient), were available to Webster.

- Jefferson: round down.
- Adams: round up.
- Dean: round down or up depending on which option gives a state's constituency closer to the divisor.
- Webster: round normally.

For apportionment based on the 1830 census Congress used the precedent Jefferson method. However, alternatives were now on the table. Apportionment based on the 1840 census used a constituency approach with the divisor 70680 and, for the first time, Webster's method for rounding the quotient. This resulted in a House with 233 members. It was the only time in U.S. history that the House size decreased as a result of the decennial census-based reapportionment process.

4 Rounding the Quotient

Step 3 in the basic divisor method involves rounding the quotient. During debates based on the 1830 census, Daniel Webster had four proposals for how to round the quotient. Jefferson's, Adams's, and Webster's rounding criteria are familiar. We now take a closer look at Dean's method with an example.

In 1830 the US population was 11,931,578.

Consider: constituency = 50,000 people.

Vermont's population: 280,657.

Vermont's quotient: $280,657/50,000 = 5.613$.

At this point, Jefferson apportions 5 seats to Vermont; Adams, 6 seats.

With 5 seats the constituency is $280,657/5 = 56,131$.

With 6 seats the constituency is $280,657/6 = 46,776$.

A constituency of 46,776 is closer to the target constituency of 50,000; hence, Dean awards Vermont 6 seats.

Dean's and Webster's methods are similar in their thinking. Webster's method involves rounding the decimal quotient normally; i.e., if the decimal fraction is less than .5, then round down, otherwise round up. Denote the quotient by q and let n be the integer part of q . Then, q rounded down is n and q rounded up is $n+1$. Rounding q normally is equivalent to the criterion: round up if and only if q is greater than the arithmetic mean of the round down, round up options; i.e., $q > AM(n, n+1)$. Dean's method is mathematically equivalent to the criterion: round up if and only if q is greater than the harmonic mean of the round down, round up options; i.e., $q > HM(n, n+1)$.⁴

The four rounding options may be expressed in terms of the quotient as follows. A state's apportionment is obtained by rounding the quotient, q , where you round up if and only if

- Jefferson: $q > \max(n, n+1)$ Since this can't happen, always round down.
- Adams: $q > \min(n, n+1)$ Since this always happens, always round up.
- Webster: $q > AM(n, n+1)$ Round normally.
- Dean: $q > HM(n, n+1)$ Round by closest constituency.

For illustration, we now apply these four basic divisor methods to the 1810 census (see Figure 6). Although there is a lot of data in the displayed spreadsheet, one can quickly grasp the main elements. First, **Census 1810** lists the 17 states with their populations. Second, Congress used a constituency approach with a congressman representing **35000** people. Third, 35000 is divided into each state's population to determine each state's **quotient**. Fourth, the quotient is rounded applying the four variations: **Jefferson**, **Adams**, **Webster**, and **Dean**. With the fixed constituency of 35000, the four rounding methods each lead to a different House size. Since Jefferson rounds all quotients down, this method produces the smallest House with 181 members. Since Adams rounds all quotients up, this method produces the largest House size with 198 members. Since Webster and Dean round some states up and other states down, they produce a House with an intermediate size. Note that Dean's method produces a House with one more member than Webster's method. The involved state is Connecticut whose quotient is 7.4805. Rounding normally, Webster rounds the quotient down and awards Connecticut 7 seats. However, Dean's rounding criterion is to round up if the quotient is larger than the harmonic mean of the round-down, round-up options. Here, $HM(7,8) = 7.466\dots$. Thus, Dean awards Connecticut 8 seats.

Census 1810		$d = 35000$				
State	Population	Quotient	Jefferson	Webster	Dean	Adams
CT	261818	7.4805	7	7	8	8
DE	71004	2.0287	2	2	2	3
GA	210346	6.0099	6	6	6	7
KY	374287	10.6939	10	11	11	11
MD	335946	9.5985	9	10	10	10
MA	700745	20.0213	20	20	20	21
NH	214460	6.1274	6	6	6	7
NJ	241222	6.8921	6	7	7	7
NY	953043	27.2298	27	27	27	28
NC	487971	13.9420	13	14	14	14
OH	230760	6.5931	6	7	7	7
PA	809773	23.1364	23	23	23	24
RI	76931	2.1980	2	2	2	3
SC	336569	9.6163	9	10	10	10
TN	243913	6.9689	6	7	7	7
VT	217895	6.2256	6	6	6	7
VA	817594	23.3598	23	23	23	24
US	6575234	188.1222	181	188	189	198

Figure 6. The 1810 census.

⁴ For the mathematical derivation of these equivalencies, see Charles Biles, *Congressional Apportionment Based on the Census 1800-1840*: 50-53; available as an open-source download from <http://www.nia977.wix.com/drbcap>.

Hence, apportionment methodology matters. The criterion for rounding the quotient matters. Although different methods can produce the same result, they may all produce different results as exhibited by apportionment based on the 1810 census.

Re-apportionment resulting from the first six censuses was accomplished using a basic divisor method. The first five re-apportionments used Jefferson's method. The sixth, based on the 1840 census, used Webster's method. However, the basic divisor method became subject to serious political manipulations and Congress looked for an alternative to eliminate gaming the system. Accordingly, in the apportionment debate based on the 1850 census Congress abandoned the basic divisor approach and applied a House size approach. This transformed the congressional apportionment problem into a mathematical apportionment problem.

Exercises

1. Consider the 1790 census.
 - A. Create a spreadsheet showing the effects of applying the four basic divisor methods (Jefferson, Adams, Webster, Dean) to the 1790 census using the divisor 33000.
 - B. Do the four methods lead to different results?
2. Repeat Exercise 1 for the 1800 census using the divisor 33000.
3. Repeat Exercise 1 for the 1820 census using the divisor 40000.
4. Repeat Exercise 1 for the 1830 census using the divisor 47700.
5. Repeat Exercise 1 for the 1840 census using the divisor 70680.
6. The state of Delaware has three counties: Kent, New Castle, and Sussex.⁵
 - A. Complete the following table to apportion the Delaware House of Representatives using the indicated basic divisor method (see Figure 6 for illustration).

Census 2010		$d = 20000$				
County	Population	Quotient	Jefferson	Webster	Dean	Adams
Kent	162310	8.1155				
New Castle	538479	26.92395				
Sussex	197145	9.85725				
Delaware	897934	44.8967				

- B. Repeat A. using a divisor of 21900.
- C. Repeat A. using a divisor or 25000.

⁵ Exercises 6 and 7 are adapted from *Mathematics and Society: A Survey of mathematics for the liberal arts major*, 2012: 76-9. Available from <http://www.opentextbookstore.com/mathinsociety/>.

7. The state of Rhode Island has five counties: Bristol, Kent, Newport, Providence, and Washington. Complete the following table to apportion the Delaware House of Representatives using the indicated basic divisor method (see Figure 6 for illustration).

Census 2010		$d = 14000$				
County	Population	Quotient	Jefferson	Webster	Dean	Adams
Bristol	49875	3.5625				
Kent	166158	11.8684				
Newport	82888	5.9206				
Providence	626667	44.7619				
Washington	126979	9.0699				
Rhode Island	1052567	75.1834				

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