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The History of the Congressional Apportionment Problem through a Mathematical Lens

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Abstract: Since the celebrated work of Balinski and Young, the mathematics community has worked to incorporate the Congressional Apportionment Problem (CAP) into a variety of university courses ranging from general education to mathematics education to mathematics in democracy courses. The CAP represents a fascinating historical and political topic with overtones of mathematical thinking. This paper gives a modern perspective to the mathematics while retaining the historical nature of events as they evolved. It also provides a general framework for classroom use in presenting and investigating the variety of apportionment methods used throughout American history.

Keywords: apportionment, Congress, census, mean, American history, congressional apportionment methods

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Charles Biles received his Ph.D. in mathematics in 1968 at the University of New Hampshire. His dissertation was in point-set topology and written under the inspiring direction of Professor Sam Shore. From 1969 to 2005 he was a professor of mathematics at Humboldt State University where he developed an interest in the mathematical modeling of renewable natural resources. After retiring from full-time teaching he began a study of American history. In a Jefferson to Jackson class he selected the congressional apportionment problem as a class research project and has been studying the topic ever since. He lives in Eureka, CA, with his dear wife Carolyn.

1. INTRODUCTION

Congressional apportionment is the process of determining how many seats in the United States House of Representatives each state gets. The Congressional Apportionment Problem (CAP) provides fascinating insight on the joint influence of history, politics and mathematics to solve a real-world problem involving multi-faceted issues of fairness and power. Factors such as House size, size of congressional districts, among others, provide material for debate. [1, 3, 9, 14] This article highlights the evolution of the basic historical and mathematical concepts of congressional apportionment from the initial census of 1790 to the present in a manner appropriate for use in general or mathematics secondary education courses. We conclude with suggestions for teaching apportionment based on a historical approach in such courses.

1.1 Prelude

A key to appreciating apportionment is to understand the concept of mean. A **mean** of a set of numbers is an average computed from them in a way that preserves some rationale for those numbers. The three most common means are the arithmetic mean (AM), geometric mean (GM), and harmonic mean (HM). For $a,b > 0$, AM = $(a + b)/2$, GM = $\sqrt{ab}$, and HM = $2/(1/a + 1/b)$. Further, $a \leq$ HM $\leq$ GM $\leq$ AM $\leq b$ with two items equal if and only if $a = b$.

1.2 Stating the CAP

The CAP is easy to state and appreciate but difficult politically and mathematically to resolve: how many seats in the House of Representatives does each state get? Figure 1 displays an overview of the current situation with the number of seats each of the 50 states was apportioned in the 435 member House based on the 2010 census.

Figure 1. The map shows the House apportionment based on the 2010 census. The apportionment population for 2010 is 309,183,463. Each congressman represents on average 710,767 people

The number of people a congressperson represents is called the **constituency**. One political issue is created by the reciprocal relationship between House size and constituency–as one decreases the other increases. The House has remained at 435 for virtually the past century.

1.3 The Constitution

Article I, Section 1, of the U.S. Constitution states that all federal legislative powers shall be vested in Congress, which shall consist of a Senate and a House. The basis for the composition of the House is described in Section 2:

- members of the House shall be chosen every second year by the people of the several states;

- representatives shall be apportioned among the states according to their respective numbers;
- the first enumeration shall be made within three years after the first meeting of Congress and subsequently every ten years;
- the size of the House shall not exceed one representative for every 30000 people, but each state shall have at least one representative.

Section 2 also specified the original House apportionment, established by the framers while creating the Constitution in Philadelphia, summer of 1787, using their guesstimates of the populations of the original 13 states. Although population increases have made the 30000 figure moot in today's world, thirtythousand.org is crusading to make 30000 the House constituency. This would lead to a House of 10283 representatives based on the 2010 census. We leave it to your imagination how such a House would function, both practically and politically.

2. THE MATHEMATICAL CAP

We now restate the CAP in mathematical language.

Let $\mathcal{U} = \{S_1,S_2,\ldots,S_s\}$ where $s \in \mathbb{N}_2$ ($\mathbb{N}_k = \{n \in \mathbb{N} \mid n \geq k\}$).
Let p_i = population of state S_i, $\vec{p} = \langle p_1,p_2,\ldots,p_s \rangle$, and $p = \sum_1^s p_i$.
Determine $\vec{a} = \langle a_1,a_2,\ldots,a_s \rangle$ where $a_i \in \mathbb{N}$.

$\mathcal{U}$ represents the federal union of s states ($s \geq 2$ to avoid trivial cases). The vector $\vec{p}$ is the census and $\vec{a}$ is the apportionment vector. Each state's apportionment must be a natural number. This constraint makes the CAP mathematically interesting. Congress has never discussed fractional representation as an option for the House. The House size is $h = \sum_1^s a_i$. The only constitutional constraint on h is that $s \leq h \leq p/30000$.

The 225 years of U.S. history has produced two general methods for resolving the CAP: divisor and quota. Divisor methods are driven by the constituency question: how many people should a congressperson represent? Quota methods are driven by the House size question: how many seats should the House have? Further, two types of divisor methods have been used: basic and modified.

The three main apportionment methods of U.S. history (basic divisor, modified divisor, quota) can be distinguished by how they treat the House size, h. In a basic divisor method, h appears only as a result of the process; h is never considered during the apportionment calculations. In a modified divisor method, h is first established but is used only as the goal. Constituency, not h, plays the dominant role in the calculations. In a quota method, h is considered as a resource and then distributed among the states. House size plays the role of goal and is used as a dominant factor in the calculations. Summarizing, in a basic divisor method, h is the result; in a modified divisor method, h is the goal; in a quota method, h is the resource for distribution.

A **quota method** contains three formal steps.

Step 1. Let $h \in \mathbb{N}_s$. (Each state gets at least one representative.)
Step 2. Calculate $Q_i = h(p_i/p)$. Let $n_i = \text{int}(Q_i)$.
Step 3. Select $a_i \in \{n_i,n_i+1\}$ so that $h = \sum_1^s a_i$.

Note how h figures in all three steps. The method is illustrated in the next section using the 1790 census.

A **basic divisor method** (BDM) also has three steps.

Step 1. Select $d \in \mathbb{N}_{30000}$.
Step 2. Calculate $q_i = p_i/d$. Let $n_i = \text{int}(q_i)$.
Step 3. Select $a_i \in \{n_i, n_i+1\}$.

Step 1 is initiated by the constituency question: how many people should a congressperson represent? The answer, d, is called the **divisor** since, as indicated in Step 2, one divides each state's population by d to determine how many representatives the state merits. The result, q_i, is called the state's **quotient**. Variations between BDMs occur in Step 3 where one decides how to round the decimal q_i.

Note how a quota method is driven by h. In contrast, a BDM is driven by d.

3. THE 1790 CENSUS

The 1st Congress authorized the first census to begin in August 1790 under the supervision of Secretary of State Thomas Jefferson. The 1st Congress also admitted Vermont as a new state. The census took over a year and was submitted to the 2nd Congress on 28 October 1791 (Figure 2: Census, Population). The numbers next to each state are the number of representatives provisionally assigned to that state.

The House immediately went to work on reapportionment. Driven by ratification debates freshly in mind, the House focused on the 30000 constituency figure, applying a BDM with $d = 30000$ (Figure 2: House Bill). The resulting quotients were rounded down, yielding $h = 112$. This process took the 67 member House (Kentucky was not yet admitted to the Union) just under a month. The House bill was then sent to the Senate for concurrence. After two weeks the Senate concurred except for d. The Senate used $d = 33000$ within a BDM and truncated the decimal quotients, resulting in $h = 105$ (Figure 2, Senate Bill, Seats).

With no agreement, an impasse ensued. Federalists tested each bill by applying the quota method to each resulting h. First, determine each state's **proportion**, p_i/p, which represents a state's fair share of the House. Then, multiply h by each state's proportion to obtain each state's **quota**, Q_i (Figure 3). An intuitive rule of fairness, the **quota rule**, is that a_i must be Q_i rounded either down or up.

Census		House Bill		Senate Bill	
State	Population	*d* = 30000	Seats	*d* = 33000	Seats
CT 5	236841	7.90	7	7.18	7
DE 1	55540	1.85	1	1.68	1
GA 3	70835	2.36	2	2.15	2
KY	68705	2.29	2	2.08	2
MD 6	278514	9.28	9	8.44	8
MA 8	475327	15.84	15	14.40	14
NH 3	141822	4.73	4	4.30	4
NJ 4	179570	5.99	5	5.44	5
NY 6	331589	11.05	11	10.05	10
NC 5	353523	11.78	11	10.71	10
PA 8	432879	14.43	14	13.12	13
RI 1	68446	2.28	2	2.07	2
SC 5	206236	6.88	6	6.25	6
VT 2	85533	2.85	2	2.59	2
VA 10	630560	21.02	21	19.11	19
US 67	3615920	120.53	**112**	109.57	**105**

Figure 2. The 1790 Census data with the first House and Senate apportionment bills.

Census		House Bill			Senate Bill		
State	Population	*d* =30000	Seats	Quota	*d* =33000	Seats	Quota
CT 5	236841	7.90	7	7.336	7.18	7	6.877
DE 1	55540	1.85	1	1.72	1.68	1	**1.613**
GA 3	70835	2.36	2	2.194	2.15	2	2.057
KY	68705	2.29	2	2.128	2.08	2	1.995
MD 6	278514	9.28	9	8.627	8.44	8	8.088
MA 8	475327	15.84	15	14.723	14.40	14	13.803
NH 3	141822	4.73	4	4.393	4.30	4	4.118
NJ 4	179570	5.99	5	5.562	5.44	5	5.214
NY 6	331589	11.05	11	10.271	10.05	10	9.629
NC 5	353523	11.78	11	10.95	10.71	10	10.266
PA 8	432879	14.43	14	13.408	13.12	13	12.57
RI 1	68446	2.28	2	2.12	2.07	2	1.988
SC 5	206236	6.88	6	6.388	6.25	6	5.989
VT 2	85533	2.85	2	2.649	2.59	2	2.484
VA 10	630560	21.02	21	**19.531**	19.11	19	**18.310**
US 67	3615920	120.53	**112**	112	109.57	**105**	105

Figure 3. Quota Rule analysis of the first House and Senate apportionment bills.

The analysis exposes a quota rule violation in the House bill for Virginia, whose fair share of 112 seats is 19.531. Hence, by quota Virginia merits either 19 or 20 seats; but, the House bill apportions 21 seats to Virginia.

The Senate bill, although satisfying the quota rule, displays an annoying feature in comparing the quotas and seats for Virginia and Delaware. The Senate bill has h = 105. Virginia's quota of 105 seats is 18.310 seats; Delaware's, 1.613 seats. Yet, the Senate bill awards Virginia 19 seats and Delaware 1 seat.

The BDM in which a state's apportion is given by the quotient rounded down is called **Jefferson's method** in today's literature. An application of Jefferson's method may suffer two flaws: quota rule violations and favoritism of one state over another. These flaws always favor large states over small states.

With the exposure of these two flaws, Congress went back to the drawing board and applied a quota method. First, using d = 30000, compute p/d = 3615920/30000 = 120.53. Then, apply the quota method with h = 120 (Figure 4). Each state's quota is a decimal. Assigning each state its lower quota distributes 111 seats. But, the goal is h = 120; hence, there are 9 more seats to be distributed among the 15 states. Congress gave these 9 seats to the 9 states with the largest fractional parts. In today's literature, this quota method is called **Hamilton's method**. In this case Hamilton's method had the coincidental good fortune that each state given the upper quota had a fractional part exceeding 0.5 and each state given the lower quota had a fractional part under 0.5.

Census		Hamilton's Method			
State	Population	h = 120	Quota	Lower Q	Appt
CT 5	236841		7.86	7	8
DE 1	55540		1.84	1	2
GA 3	70835		2.35	2	2
KY	68705		2.28	2	2
MD 6	278514		9.24	9	9
MA 8	475327		15.77	15	16
NH 3	141822		4.71	4	5
NJ 4	179570		5.96	5	6
NY 6	331589		11.00	11	11
NC 5	353523		11.73	11	12
PA 8	432879		14.37	14	14
RI 1	68446		2.27	2	2
SC 5	206236		6.84	6	7
VT 2	85533		2.84	2	3
VA 10	630560		20.93	20	21
US 67	3615920	120.531	120	111	120

Figure 4. The first apportionment bill passed by Congress.

This resulted in the first apportionment bill passed by Congress. On 26 March 1792, five months since Congress received the 1790 census, the bill was sent to President Washington for approval. Washington vetoed the bill. This veto is significant for three reasons:

- it was the first presidential veto in U.S. history;
- it was the only veto of Washington's first administration;
- Washington justified his veto based on his interpretation of the Constitution.

The House size of 120 yields 3615920/120 = 30133 when applied to the U.S. population as a whole. But, when applied to Connecticut, 236841/8 = 29605. Washington insisted that the constitutional constraint that h shall "not exceed one for every thirty Thousand" must be satisfied by each state.

After Washington's veto, Congress quickly passed the original Senate bill. Jefferson's method, used to create the Senate bill, set precedent and was used for the next five censuses (Figure 5).

1790: s = 15, d = 33000 $\Rightarrow$ h = 105
1800: s = 16, d = 33000 $\Rightarrow$ h = 141
1810: s = 17, d = 35000 $\Rightarrow$ h = 181
1820: s = 24, d = 40000 $\Rightarrow$ h = 213
1830: s = 24, d = 47700 $\Rightarrow$ h = 240
1840: s = 26, d = 70680 $\Rightarrow$ h = 223

Figure 5. Jefferson's method applied to the first six censuses.

Flaws with Jefferson's method were evident from the start, but new quota rule violations demanded attention. Alternate proposals for rounding the decimal quotient surfaced. During the 1830 census-based apportionment debates Daniel Webster, chair of the Senate apportionment committee, received letters from John Quincy Adams, a representative from Massachusetts, and James Dean, a mathematics professor at the University of Vermont. Thinking about alternatives proposed by Adams and Dean, Webster devised his own. Thus, four variations of the BDM, all dealing with how to round a decimal, were available to Webster.

- Jefferson: round down.
- Adams: round up.
- Dean: round down or up depending on which option gives a state's constituency closer to the divisor.
- Webster: round normally.

Adam's method was never seriously considered. It suffered from the same flaws as Jefferson's method; in particular, it was subject to quota rule violations and it could show bias. The Adams bias was always in favor of small states over large states.

4. DEAN'S AND WEBSTER'S METHODS

These methods are variations of the 3-step BDM:

Step 1. Select $d \in \mathbb{N}_{30000}$.

Step 2. Calculate $q_i = p_i/d$. Let $n_i = \text{int}(q_i)$.

Step 3. Select $a_i \in \{n_i, n_i+1\}$ where $a_i = n_i+1$ if and only if

Dean: $p_i/(n_i+1)$ is closer to d than p_i/n_i.

Webster: $q_i > n_i + 0.5$.

A numerical example clarifies Dean's method. Consider $d = 50000$. In 1830, $p_{\text{VT}} = 280657$. So $q_{\text{VT}} = 280657/50000 = 5.6$. At this point Jefferson apportions Vermont 5 seats; Adams, 6 seats. With 5 seats, Vermont's constituency is 280657/5 = 56131; with 6 seats, 280657/6 = 46776. Since 46776 is closer to the target d = 50000 than 56131, Dean awards Vermont 6 seats.

Think of d as the constituency target (Figure 6). Now $n_i = \text{int}(q_i)$ is the largest natural number such that $p_i/n_i > d$ and n_i+1 is the smallest natural number such that $p_i/(n_i+1) < d$. With d as the target and apportionment as a natural number, p_i/n_i and $p_i/(n_i+1)$ are the two best shots that a BDM can take at d. Dean merely asks, which shot comes closer to the target?

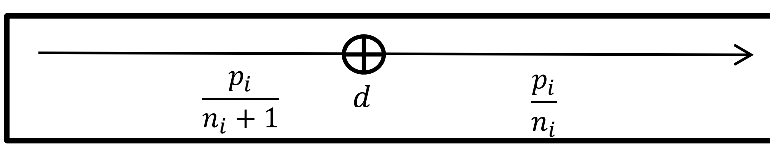

Figure 6. A graphic of Dean's Method.

Hence, Dean's criterion yields

$$a_i = n_i + 1 \iff d - \frac{p_i}{n_i+1} < \frac{p_i}{n_i} - d \iff \frac{2}{\frac{1}{n_i}+\frac{1}{n_i+1}} < \frac{p_i}{d} \iff \text{HM}(n_i, n_i+1) < q_i .$$

Dean's reasonable rounding criterion is mathematically equivalent to round up if and only if the quotient is greater than the harmonic mean of the options n_i and n_i+1.

Webster's method (round normally) can be presented by remodeling Dean's target range (Figure 6) with a reciprocal plan (see Figure 7). Where Dean focuses on the representative (congressperson, how many people do you represent?), Webster focuses on the individual (citizen, how many representatives do you have?). This can be seen from the units of the target:

- d people/representative (constituency)
- $1/d$ representatives/person

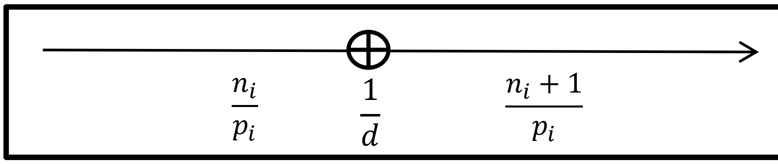

Figure 7. A graphic of Webster's Method.

Hence, Webster's criterion yields

$$a_i = n_i + 1 \iff \frac{n_i+1}{p_i} - \frac{1}{d} < \frac{1}{d} - \frac{n_i}{p_i} \iff \frac{n_i+(n_i+1)}{2} < \frac{p_i}{d} \iff \mathrm{AM}(n_i, n_i+1) < q_i .$$

The enlightening point here is that Webster's reasonably sounding arithmetical rounding criterion is mathematically equivalent to round according to which option, n_i or $n_i + 1$, is closer to the targeted individual representation, in contrast to Dean's targeted constituency. Ironically, Dean, the mathematics professor, was unaware of the mathematical (harmonic mean) equivalency of his political proposal, while Webster, the consummate politician, was unaware of the political equivalency of his arithmetical proposal (round normally).

5. THE 1840 CENSUS

Alternate proposals for rounding the decimal quotient during the 1830 census reapportionment debate were just ideas within a Senate committee. The precedent Jefferson method was in effect for 50 years and Congress, as a whole, was not interested in new methods of apportionment. The chair of the House apportionment committee, James K. Polk (Tennessee), brilliantly manipulated Jefferson method's and steered the political action in Congress to passage of the 1830 census-based apportionment bill that applied the intriguing divisor 47700. Polk, who was good at mathematics, realized that the resulting quotient and applied divisor are inversely related to each other. By decreasing the divisor, a state's quotient may increase just enough to yield an extra congressperson. For example, in 1830 the change of divisor from 48000 to 47700 changed Georgia's quotient from 8.954 to 9.011. In this manner Polk's micromanagement resulted in an extra seat in the House for Georgia, Kentucky and New York, states politically important to President Andrew Jackson. During Jackson's second term Polk became Speaker of the House. Later, New York became the deciding state in the election of James K. Polk as eleventh president of the U.S. To this day Polk is the only Speaker of the House subsequently elected President.

As the result of Polk's manipulations became clear during the decade of 1830, congressmen prepared themselves. The apportionment debate based on the 1840 census began with a political game of divisor. On 2 April 1842 in the 242 member House, 59 motions were made to establish d. Values ranged from 30000 to 141000 with the majority ranging from 50159 to 62172. The House passed an apportionment bill using a BDM with d = 50179 resulting in h = 305. The bill was then sent to the Senate for concurrence.

The bill would increase the House by 63 members, the largest increase ever. Senators were already concerned that h = 242 was too large. The press referred to the House as the "bear garden." A Senate committee blanked the House's divisor and then engaged in its own divisor game. On 26 May there were 27 motions for a divisor with values ranging from 49594 to 92000. All but two proposed a divisor greater than the House value. Although motivated by House size, the Senate did not address the issue directly by specifying h, but chose to control h by specifying d. After two weeks the Senate agreed to 70680, a divisor proposed by James Buchanan (later elected 15th President of the U.S.).

Apportionment based on the 1840 census used a BDM with d = 70680 resulting in h = 233 and, for the first time in U.S. history, Webster's method of rounding. This is the only time in U.S. history that h decreased as a result of the decennial census-based reapportionment process.

6. THE 1850 CENSUS

Representative Samuel Vinton (Whig, Ohio) wanted to remove partisan wrangling by establishing an apportionment procedure before the census was taken. The Vinton Act of 1850 was a major step in the evolution of apportionment history. The act addressed the issue of House size directly, specifying h = 233 seats apportioned by Hamilton's method. The method was nominally used for the next 60 years—nominal since Congress enacted supplements which countered Hamilton's method. Such alterations were significant, possibly resulting in the selection of Rutherford B. Hayes as the 19th president over Samuel Tilden. Even though Tilden won the popular vote, Hayes won the electoral vote 185-184. If the Vinton Act were followed, Tilden would have won the electoral vote. Perhaps not surprisingly, Hayes was known during his single term as "Rutherfraud" Hayes. But experience from history exposed deeper problems that doomed the quota method.

The quota method displayed counter-intuitive paradoxes, especially the Alabama paradox that became prominent after the 1880 census: when h is increased, a state's apportion may decrease. The results for the 1900 census doomed Hamilton's method. In particular, Maine's apportion oscillated wildly: 3 seats for h = 350-382, 386, 389-390 but 4 seats for h = 383-385, 387-388, 391-400. Colorado received 3 seats in each case except h = 357 where Colorado received 2 seats. Other paradoxes, such as the population growth paradox and the new states paradox, also surfaced. The population growth paradox states that a faster growing state may lose a seat to a slower growing state; at worst, a state that grows in population may lose a seat to a state that declines. The new states paradox states that if a new state is added to the union and given its share of representatives, then upon recalculation based on the new House size, apportionments of other states may change. These unintuitive paradoxes can only be appreciated by looking at numerical examples (see references for resources).

Congress coped with such emerging problems after the 1880 and 1890 censuses by choosing a House size in which no state would lose a seat and for which Hamilton's and Webster's methods agreed. The result yielded h = 325 in 1880 and h = 356 in 1890.

7. MODIFIED DIVISOR METHODS

Apportionment based on the 1900 census came from combining methodologies. Congress started with Hamilton's method for h = 384, then rounded up all quotas whose fractional remainder was greater than .5. Hence, Hamilton's method initiated the process; but, Webster's rounding method completed it resulting in h = 386.

In 1910 Congress abandoned the quota method. For the first time in U.S. history a **modified divisor method** (MDM) was explicitly used. An MDM uses five steps:

Step 1. Select $h \in \mathbb{N}_s$.

Step 2. Select $d \in \mathbb{N}_{30000}$.

Step 3. Calculate $q_i = p_i/d$. Let $n_i = \text{int}(q_i)$.

Step 4. Let $a_i = \text{round}(q_i) \in \{n_i, n_i+1\}$.

Step 5. If $\sum_1^s a_i = h$, then DONE; ELSE, modify d and GO TO Step 3.

Although one selects h in Step 1, h is used only as the goal in Step 5. The actual calculations are based on the divisor selected in Step 2. Variations to the MDM occur in Step 4 where one chooses a method for rounding (e.g., Jefferson, Adams, Dean, or Webster).

Based on the 1910 census, Congress specified $h = 433$ (the smallest house size for which no state lost a seat) and used Webster's method of rounding. Further, Congress stipulated one seat each for Arizona and New Mexico upon admission to the Union, which happened in 1912.

However, there was so much confusion and gridlock politics from the 1920 census that for the only decade in U.S. history no census-based reapportionment was made. Congress could not agree on a method for apportionment; in particular, how to round the decimal quotients. Gridlock politics centered on prohibition—the dries, favoring retention, would not agree to anything that gave the wets, favoring abolition, more power.

In response to the non-reapportionment from the 1920 census, newly elected President Herbert Hoover called a special session of Congress. The result was the Apportionment Act of 1929 which froze the House size at 435. In response to the 1930 census, Congress used $h = 435$ and applied the Webster MDM (which fortunately agreed with a competing method).

The method currently in use is described in Title 2 of the U.S. Code consisting of the Apportionment Act of 1929 along with its 1940 and 1941 amendments. The 1941 amendment, signed by President Franklin Roosevelt, fixed Huntington-Hill as the apportionment method. Important features of Title 2 include that $h = 435$ and the Huntington-Hill method will be used in the future until changed by Congress.

As an MDM, Huntington-Hill is a 5-step method.

Step 1. $h = 435$.

Step 2. Select d.

Step 3. Calculate $q_i = p_i/d$. Let $n_i = \text{int}(q_i)$.

Step 4. Select $a_i \in \{n_i, n_i+1\}$ where $a_i = n_i + 1$ if and only if $q_i > \text{GM}(n_i, n_i + 1)$.

Step 5. If $\sum_1^s a_i = h$, then DONE; ELSE, modify d and GO TO Step 3.

To understand the thinking behind the rounding method, review the graphic of Dean's method (Figure 6). Recall that Webster's method may be understood as the reciprocal remodel of Dean's (see Figure 7). For Huntington-Hill we construct another remodel of Dean's graphic: divide Dean's quantities by d, yielding a dimensionless target of 1 (Figure 8).

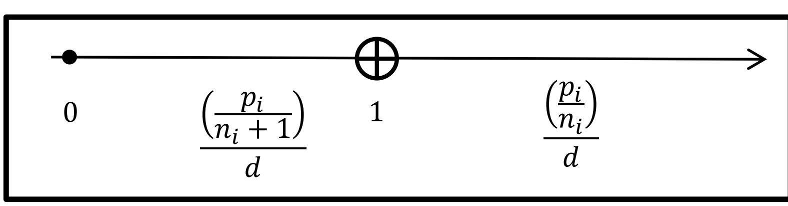

Figure 8. A graphic of the Huntington and Hill method.

If we measure which option is closer by using the linear distance measure used for Dean and Webster, we only reinvent Dean's method. Recall: if $0 < x < 1$, then $1 < 1/x$. Then round the quotient up if and only if the reciprocal of the left fraction is less than the right fraction; hence,

$$a_i = n_i + 1 \iff \frac{d}{\left(\frac{p_i}{n_i+1}\right)} < \frac{\left(\frac{p_i}{n_i}\right)}{d} \iff \sqrt{n_i(n_i+1)} < q_i \iff \mathrm{GM}(n_i, n_i + 1) < q_i\,.$$

The enlightening point here is that the Huntington-Hill rounding procedure is equivalent to round the quotient according to which shot is "closer" on a dimensionless shooting range.

8. AFTERMATH

In 1982 Balinski and Young proved that there are no problem-free apportionment methods. Any quota method is subject to the Alabama paradox and any divisor method can violate the quota rule. Although perfection is not possible, improvement is.

The Wyoming Rule, a recent reform proposal, is a BDM in which the divisor is the population of the least populous state (currently Wyoming; hence, the name). The average constituency of a congressperson based on the 2010 census is 710767. Wyoming has an apportionment population of 568300 and 1 congressman. California has an apportionment population 66 times that of Wyoming and 53 representatives. The Wyoming Rule corrects this discrepancy. However, being a BDM, House size fluctuates. Using Hill's rounding method, based on the 2000 census the Wyoming Rule produces $h = 568$; on the 2010 census, $h = 542$.

Apportionment is one concern in the general topic of representation. Usually other topics appear more pressing; e.g., gerrymandering, the census (who is enumerated), suffrage (who may vote), or the structure of the ballot. Despite these other justified issues, apportionment is a basic concern of fairness. The struggle between fairness and power is a major component of political activity affecting apportionment decisions. Accordingly, apportionment will remain a topic for mathematical, historical and political analysis.

8.1 Priority Techniques

The Census Bureau uses a different calculation technique than this paper. The Bureau first assigns one seat to each state. Then priority numbers are calculated for each state to determine who gets the next seat. For example, in the Huntington-Hill method, the priority number for a state to receive a 2^{nd} seat is obtained by dividing the state's population by GM(1,2); for a 3^{rd} seat, GM(2,3); for an n^{th} seat, GM(n-1,n). Similarly, priority numbers using Dean's method are obtained by dividing a state's population by the corresponding harmonic mean; Webster's method, the corresponding arithmetic mean. However, priority methods are mathematically equivalent to those presented in this paper.

9. CLASSROOM CONNECTIONS

The CAP provides undergraduate students in general and secondary education classes with a phenomenal opportunity to view the historical development of congressional representation through a mathematical lens. Although a seemingly straightforward mathematical task, apportionment becomes complex once one considers potential divisors, House sizes, and rounding methods (let alone issues related to politics, power and fairness)!

When presenting this problem to undergraduate students, we found that at least two periods of instruction work best. This allows the instructor to provide the historical context for this problem in Day 1 and promote active student engagement with apportionment throughout Day 2. We have done these lessons within both 50-minute and 75-minute class periods.

9.1 Day 1

To introduce this problem, students are provided with a brief reminder of mean and the various methods for computing and thinking about mean. In general terms, selecting AM, GM or HM is determined by the problem situation and some rationale of the numbers that one wants to preserve. Whereas AM is an additive mean, GM is multiplicative and HM is composite since $HM = (GM)^2/AM$. Providing students with an example of each is beneficial. Though at first this decontextualized reminder of mean may seem contrived, it provides students with a similar starting point and foundation for the entire CAP situation. Additionally, the historical narrative is much more engaging and effective once the mathematical framework has been set!

As a transition into the historical framework for this problem, students are then encouraged to keep the following perspectives in mind:

- What was it like to live back then?
- How did we get from then to now?

To appreciate a topic in history, students must resist the temptation to take today's values and norms and project them into the past. Judgments and assumptions related to race, culture, society and social norms will cloud students' ability to predict history and engage apportionment with a historical perspective.

Then the instructor introduces the CAP and its historical basis in Article I, Sections 1 and 2, of the U.S. Constitution. Connections to the most recent census provide students with a real-world context that highlight p, d and h. Initial discussions of the CAP include a description of provisional apportionment for the Second Congress and subsequent national and state populations determined during the first census. Without providing any type of introduction to either divisor or quota methods, students are reminded of constitutional guidelines and shown the population of the first 15 states (see Figure 2, Census).

Names of all congressmen from 1792 are placed on individual slips of paper and put into a hat (a complete listing of representatives from 1792 can be found online). A student draws a name and then takes on the role of that congressman to determine an apportionment plan for the next Congress. Their homework assignment includes briefly investigating the life of their congressman and drafting an apportionment plan while keeping in mind "What was it like to live back then?" (Figure 9).

9.2 Day 2

In a whole class discussion, students are asked to share the various House sizes they came up with as part of their apportionment proposals. Based on past experience with these materials,proposed values of h ranged from the low 50s to 130 and beyond. The instructor should record student responses on the board and use the visible discrepancies to prompt the creation of congressional committees. With the 1790 population table projected (see Figure 2) students self-select into groups of 3 or 4 and are instructed to reconcile their individual bills from each committee member into one bill. A committee chairperson is selected and, after about 10 minutes of negotiation, each chairperson reports the House size of the reconciled bill. Their proposed House size should also include a brief description of their apportionment process.

Put yourself in the shoes of the congressman whose name you drew. Activate the buttons "What was is like to live back then," and "How did we get from then to now?" Feel free to use the internet to research your congressman and information relevant to his decision making process using websites such as: http://bioguide.congress.gov/biosearch/biosearch.asp or http://en.wikipedia.org/wiki/2nd_United_States_Congress. However, DO NOT look up the final method used after the 1790 census (or any method since then)!
Submit a paragraph about yourself (i.e., the congressman: who you are, where you are from, your political leanings and any other relevant information). Then, use the guidelines in the Constitution, the only guidelines available, construct an apportionment bill to propose in Congress. That is, how should the seats in the House of Representatives be (re)allocated? How many seats does each state get? How many seats does your state get? What is the total number of seats in the House based on your apportionment plan? How does the power associated with your apportionment proposal balance with the power structure of the Congress?

Figure 9. CAP homework instructions for students

Previous use of these materials has repeatedly demonstrated that:

- different apportionment committees will offer different bills;
- students become actively involved in the apportionment process through the development of their own proposal and subsequent negotiation with other committee members;
- students take to heart the directives to put themselves in their congressman's shoes and evaluate this problem from a historical context;
- this active problem solving process effectively illustrates the depth of the apportionment problem to the students in the class; and
- as with most interesting mathematical problems, there are a wide variety of different solutions to the same problem.

This activity naturally leads into a discussion on how the first Congress solved the problem. The instructor highlights the mathematical development of the House and Senate Bills of 1792 (Figure 3), Jefferson's Method, Hamilton's Method (Figure 4), and Washington's subsequent veto. While going through these processes, it is worthwhile to stop and ask students for similarities they see between their own apportionment proposals, their subcommittee proposal, and the various historical methods developed.

We have found that students' investigations into apportionment, as part of their Day 1 homework, naturally lead into the mathematical development of other methods (e.g., Dean's and Webster's) and potential paradoxes (e.g., Alabama). Additionally, students are intrigued by the political manipulations of Polk and the remedies of the Vinton Act and Huntington-Hill Method. The activity proposed in this article has been developed in a way that instructors can adapt the historical narrative provided above to develop a classroom ready presentation that builds on students' work and is rich in both history and mathematics. For additional support, please feel free to contact the authors or refer to Balinski and Young's seminal text.

9.3 Common Student Solutions

Over the course of several implementations of the CAP in undergraduate general and mathematics secondary education courses, patterns in student responses have arisen. So far in our experience, every student applied a basic divisor method to construct an apportionment proposal. Common divisors are *d* = 30000 or *d* = 55540. The divisor 55540 is the apportionment population of Delaware, the smallest state in the 1790 census. This anticipates the Wyoming Rule. Students like it when the professor relates later apportionment methods to the work they did in their assignment.

Students quickly find that there are issues with the decimal portion of the quotient and with fair representation. To resolve the decimal issue, students round normally (Webster), up (Adams), or down (Jefferson). Students whose apportionment proposals are concerned with "fairness" are usually motivated by political concerns. Most of these students drew a congressman from a small state and were concerned that their small state not be overwhelmed by large states. Such concerns duplicate those debated in the Constitutional Convention in which the Great Compromise leading to our bicameral legislature was formulated.

Students also invent unique interpretations and formulate challenging questions. Unique interpretations of the 30000 constitutional constraint on the size of the House are common in student apportionment proposals for the 1790 census. These interpretations form a reference point in discussing President Washington's veto of the first apportionment bill. Some questions defy satisfactory answers; for example, why is the House size of 415 used today?

10. CLOSING REMARKS

The Congressional Apportionment Problem (CAP) deals with the equitable distribution of congressional seats following each U.S. census. While this problem represents a fascinating historical topic in its own right, the underlying mathematics of the CAP provides an intriguing investigation into the mathematical implications of political and historical factors such as size of the House of Representatives, size of representational constituency, number of states, states' populations, politics, power, and fairness. The resulting problem solving and mathematical modeling provide fertile ground on which to develop undergraduate mathematics students' sense of what it means to "do" mathematics, why and how we use mathematics to solve real-world problems and the historical contexts of mathematical connections to their everyday lives.

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