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Teaching Apportionment

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Charles Biles (cmb2@humboldt.edu) received his Ph. D. in mathematics from the University of New Hampshire under the direction of Samuel Shore. Subsequently he joined the mathematics faculty at Humboldt State University, a position which he enjoyed for 37 years. He is an avid student of U. S. history and became interested in congressional apportionment as a research project in an American history course. His ongoing project is a decade-by-decade study of congressional apportionment.

An objective of several university courses is to present a variety of current-interest topics that utilize mathematical thinking. One such topic is apportionment defined as the process of distributing a fixed number of indivisible resource units among competing groups according to some measurable group asset. A featured application is congressional apportionment: how many seats in the U. S. House of Representatives should each state get based on the decennial census and constitutional guidelines [**3**], [**6**], [**7**], [**10**]. Congressional apportionment has two different approaches: constituency and House size. The constituency approach starts with the question, how many people should a congressperson represent? The House size approach starts with the question, how many seats should there be in the House? The constituency approach was used for re-apportionment of the House based on the census years 1790-1840 [**1**], [**2**], [**4**]. However, the constituency approach does not lead to a fixed resources distribution problem. Hence, most mathematics texts contort the colorful history of congressional apportionment based on the first six censuses by forcing it into the House size approach which does yield a fixed resources distribution problem. This results not only in errors in portraying the historical record, but also in a missed opportunity to present a rather dazzling application of some really basic problems such as how to average two numbers and how to round a decimal.

An average lesson

To set the mathematical props on the stage of congressional apportionment, a class lecture should be devoted to two basic tasks: averaging and rounding. Suppose that $0 \le a < b$. What is the average of a and b? American history of congressional apportionment supplies five answers to this question [**2**]. Denote the average of a and b by ave(a,b). Then, ave(a,b) =

1. max(a,b) — the maximum of a and b
2. min(a,b) — the minimum of a and b
3. AM(a,b) = (a + b)/2 — the arithmetic mean of a and b

4. $\mathrm{HM}(a,b) = 2ab/(a + b)$ the harmonic mean of a and b
5. $\mathrm{GM}(a,b) = \sqrt{ab}$ the geometric mean of a and b

Each of these averages can be applied to the problem of how to round a decimal. Suppose $q > 0$ with integer part n where $q - n > 0$. Denote the rounding of q by round(q). Then round(q) $\in \{n, n+1\}$ where round(q) $= n + 1$ if and only if

1. $q \geq \max(n,n+1)$ round down since this criterion is never satisfied
2. $q \geq \min(n,n+1)$ round up since this criterion is always satisfied
3. $q \geq \mathrm{AM}(n,n+1)$ round normally
4. $q \geq \mathrm{HM}(n,n+1)$ harmonic mean rounding
5. $q \geq \mathrm{GM}(n,n+1)$ geometric mean rounding

The basic divisor method

Let $U = \{S_1,S_2,\ldots,S_N\}$ be a federal union of N states (N is a natural number, $N\geq 2$). Let $\langle p_1,p_2,\ldots,p_N\rangle$ denote the census; i.e., p_i is the population of state S_i. The congressional apportionment problem is to determine an apportionment vector $\langle a_1,a_2,\ldots,a_N\rangle$ where each a_i is a natural number. The census is necessary to follow the constitutional guideline that apportionment among the states be "according to their respective numbers" as enumerated by a decennial census. A constituency approach to congressional apportionment naturally leads to the basic divisor method which applies a 3-step algorithm.

Step 1. How many people should a congressperson represent? Answer: d.
Step 2. Calculate each state's **quotient**: $q_i = p_i/d$.
Step 3. Let $a_i = \max(\mathrm{round}(q_i),1)$.

Step 3 is formulated to satisfy the constitutional mandate that each state receive at least one seat in the House. Each apportionment act based on the censuses from 1790 through 1840 used this 3-step algorithm. The acts from 1790-1830 rounded the quotient by rounding down. During the debates based on the 1830 census, three additional alternatives were proposed: John Quincy Adams, round up; James Dean, round up if and only if $p_i/(n_i+1)$ is closer to d than p_i/n_i; Daniel Webster, round normally. Dean's proposal is mathematically equivalent to the harmonic mean rounding criterion while Webster's proposal is equivalent to the arithmetic mean rounding criterion [**1**], [**2**]. The apportionment act based on the 1830 census continued tradition by rounding down. The act based on the 1840 census rounded normally. Hence, by the time of the apportionment act based on the 1840 census there were four variations of the basic divisor method. These variations, each essentially concerned with how to round a decimal, are identified with a historical reference as follows.

Jefferson: round down.
Adams: round up.
Webster: round normally (use the arithmetic mean criterion).
Dean: round using the harmonic mean criterion.

The quota method

Note that the House size is merely the result of the basic divisor method; hence, a constituency approach to congressional apportionment does not lead to a fixed resource distribution problem. Thus the historic congressional apportionments based on the censuses 1790-1840 are not applications of apportionment as defined in modern texts. The first apportionment act to apply the fixed resource distribution definition was based on the census of 1850 which set the House size, h, at 233. After setting h Congress applied the quota method, a method based on the natural premise that if a state has x% of the population, then it should have x% of the seats in the House. The quota method utilizes a 4-step algorithm.

Step 1. Determine the House size, h.
Step 2. Calculate each state's **quota**: $Q_i = h(p_i/p)$ where $p = \sum p_i$.
Step 3. Let L_i be the integer part of Q_i. Initialize $a_i = L_i$.
Step 4. Create a priority list to distribute the remaining $h - \sum L_i$ seats.

The quota represents a state's "fair share" of h seats based on its share of the national population, p. Invariably, Step 3 distributes most but not all of the seats and one is faced with the situation that $0 < h - \sum L_i < N$. The remaining $h - \sum L_i$ seats are distributed by means of a priority list. American history has offered the following options for this priority list [**1**], [**2**].

Hamilton: $Q_i - L_i$.
Lowndes: p_i/L_i.
Hill: $p_i/\mathrm{GM}(L_i, L_i + 1)$.

Hamilton's quota method is the only variation in American history ever applied to formulate an apportionment act based on the quota method.

The modified divisor method

Congress abandoned the basic divisor method after the apportionment act based on the 1840 census primarily because the method suffered from rampant political gamesmanship. Congress abandoned the quota method after the discovery of deal-breaking paradoxes, especially the Alabama Paradox [**1**]-[**4**], [**6**]-[**9**]. The basic divisor method is based on the constituency approach to congressional apportionment while the quota method is based on the House size approach. Since these are the only two approaches to the congressional apportionment problem, Congress sought to blend the two methods in a way that would avoid their worst results. Accordingly, Congress adopted the modified divisor method which utilizes a 5-step algorithm.

Step 1. Determine the House size, h.
Step 2. Initialize the divisor d with p/h (p is the national population).
Step 3. Calculate each state's quotient: $q_i = p_i/d$.
Step 4. Let $a_i = \max(\mathrm{round}(q_i),1)$.
Step 5. IF $\sum a_i = h$, THEN DONE; ELSE modify d and GO TO Step 3.

The modified divisor method is merely the basic divisor method with a predetermined answer. Textbooks refer to the initial divisor calculated in Step 2 as the standard divisor [**3**], [**6**], [**7**], [**10**]. One calculates the standard divisor as a reasonable value to initiate the divisor algorithm; however, it usually does not produce the desired House size, h, in Step 5. Accordingly, this value for d must be adjusted (modified) in order to obtain the specified value for h.

Variations occur in Step 4 where one must choose a rounding technique. In addition to the four rounding techniques inherited from the basic divisor method, another was introduced during discussions based on the 1910 census. Edward Huntington advocated the rounding technique based on the geometric mean, the same criterion Joseph Hill used to create a quota method priority list. Accordingly, this variation is called the Huntington-Hill method.

Many of today's mathematics writings refer to Jefferson's, Adams's, Dean's, Webster's, and Huntington-Hill's methods only in the context of a modified divisor method [**3**], [**6**], [**7**], [**8**], [**10**]. These adjectives only specify the rounding technique and can serve this purpose for both basic and modified divisor methods. It is noteworthy that current congressional apportionment law specifies the Huntington-Hill modified divisor method with h = 435 [**4**].

Priority techniques

The modified divisor method accomplishes the goal of avoiding the worst problems of the basic divisor method and the quota method. However, the modified divisor method was presented applying an ad-hoc algorithm specific to a given House size. If one wants to compare the results with other House sizes, then one needs to re-run the algorithm for each size of interest. Accordingly, the Census Bureau developed a serial technique for distributing seats in the House [**4**]. First, each state is given one seat each. This complies with the constitutional requirement that each state must have at least one seat. The Constitution further specifies that House seats are to be based on population. Today, giving one seat to each state distributes 50 seats. In a serial approach for further distribution, we ask, which state has priority for the 51^{st} seat? 52^{nd} seat? 53^{rd} seat? Etc. In general, if a state has n seats, what is its priority for gaining an additional seat?

In response, let PN(n) be the priority number for a state to receive an $(n+1)^{st}$ seat given that the state has n seats. We define PN(n) = p_i/ave(n,n+1). We then achieve each of the five modified divisor methods by setting ave(n,n+1) as follows [**1**], [**2**], [**4**].

Jefferson:	max(n,n+1)
Adams:	min(n,n+1)
Webster:	AM(n,n+1)
Dean:	HM(n,n+1)
Huntington-Hill:	GM(n,n+1)

Today the Census Bureau calculates the priority values for seats 51 through 440 using the Huntington-Hill method. Since current law specifies a 435 seat House, based on the 2010 census seat 434 went to California, seat 435 to Minnesota, and seat 436 would have gone to North Carolina [**5**].

The classroom

The congressional apportionment problem is a magnificent problem to incorporate not only into liberal arts mathematics courses but also secondary education teacher training courses. A key point of this paper is that the American history of this problem acts as a driver and motivator for the mathematics. Accordingly, using the standard 50-minute class length as a model, it works well to devote five days to apportionment as follows.

Day 1. An average lesson.
Day 2. The basic divisor era: 1790-1840.
Day 3. The quota method.
Day 4. The modified divisor method.
Day 5. Priority computation techniques.

Open-source materials for these topics are available on the author's website [**2**]. Day 1 establishes the skills needed for apportionment calculations. It also leaves the student with a "what's this stuff good for?" feeling that is satisfied in Days 2-5 where the five averaging and rounding mechanisms are applied to a real problem in American history. Day 2 focuses on the basic divisor method which establishes the platform for studying fixed-resources distribution problems. Congressional apportionment serves to motivate the evolution of mathematical thinking about apportionment rather than merely serving up examples.

Epilogue

The congressional apportionment problem is easy to state but challenging to resolve. Resolution first requires a choice of approach: constituency or House size. The constituency approach naturally led to the basic divisor method. The House size approach first led to the quota method and then to the modified divisor method. Divisor methods introduced the problem of how to round a decimal. These methods subsequently introduced the challenge of how to create a priority list. At the foundation is the question of how to average two numbers.

Although averaging two numbers and rounding a decimal may sound trivial at first, they lead to substantial situations demanding in-depth analysis, making apportionment an ideal liberal arts topic. The depth of the subject is portrayed by the stunning Balinski-Young Impossibility Theorem that there are no perfect apportionment methods: any divisor method is subject to quota violations and any quota method is subject to paradoxes [**1**]. Accordingly the Balinski-Young Theorem is to apportionment what Arrow's Theorem is to voting theory.

Finally, one may want to conclude a presentation of apportionment with a view to the future since some change in current law is inevitable. Possible reform ideas include the Wyoming rule, the proposals of thirtythousand.org, and the proposal of

Neubauer and Gartner [9], or simply replacing the Huntington-Hill criterion for rounding by Webster's [1], [2].

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Summary. The history of congressional apportionment serves well as background and motivation for a comprehensive treatment of apportionment. American history of congressional apportionment presents two approaches: constituency and House size. These approaches produced the Jefferson, Adams, Dean, Webster, and Huntington-Hill divisor methods along with the Hamilton, Lowndes, and Hill quota methods. Many mathematics textbooks, however, treat apportionment solely as a fixed resources distribution problem, thereby ignoring the constituency approach resulting in errors in presenting the historical record.

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